Equality of Power and Fair Public Decision-making

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Abstract. Ronald Dworkin's *equality of resources* [12], and the closely related concept of envy-freeness, are two of the fundamental ideas behind fair allocation of private goods. The appropriate analog to these concepts in a public decision-making environment is unclear, since all agents consume the same "bundle" of resources (though they may have different utilities for this bundle). Drawing inspiration from equality of resources and the Dworkin quote below, we propose that equality in public decision-making should allow each agent to cause equal cost to the rest of society, which we model as equal externality. We term this *equality of power*. The first challenge here is that the cost to the rest of society must be measured somehow, and it is generally impossible to elicit the scale of individual utilities (in the absence of monetary payments). Again drawing inspiration from foundational literature for private goods economies, we normalize each agent's utility so that every agent's marginal utility for additional power is the same. We show that for quadratic utilities, in the large market limit, there always exists an outcome that simultaneously satisfies equal power, equal marginal utility for additional power, and social welfare maximization with respect to the normalized utilities.

> "*Equality of resources supposes that the resources devoted to each person's life should be equal. That goal needs a metric. The auction proposes what the envy test in fact assumes, that the true measure of the social resources devoted to the life of one person is fixed by asking how important, in fact, that resource is for others. It insists that the cost, measured in that way, figure in each person's sense of what is rightly his and in each person's judgment of what life he should lead, given that command of justice.*"
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> Ronald Dworkin, *What is Equality? Part II: Equality of Resources*, 1981

1 Introduction

In settings where monetary payments are not allowed, it is generally impossible to elicit the absolute scale of agents' utilities. This makes objectives like social welfare maximization difficult. Instead, it is common to focus on some notion of equality or fairness. In the context of pure private goods economies, this is commonly represented, in both analytic philosophy and economics, by the closely related ideas of the envy-freeness [15], competitive equilibrium from equal incomes [26] and equality of resources [12]. There has been a recent surge of interest in these topics – and more generally, axiomatic fair division of resources – in the computational economics community as well.

It is not immediately clear how to adapt these concepts to the public decision-making setting. For example, envy-freeness is not meaningful in such an economy, since all agents "consume" the same outcome; they simply receive different utility from that outcome. In this paper we propose and analyze a potential solution in a continuous public decision-making environment (i.e., an outcome is a point in $\mathbb{R}^m$, where each of the m dimensions represents a public issue) that we call *equality of power*.

The idea at the foundation of equality of power is that each individual's opinion should be given equal weight. This is widely considered by political theorists to be the defining feature of democracy [11] if not of justice more generally [1, 2, 25]. As Dahl puts it, "The moral judgement that all human beings are of intrinsically equal worth...(requires) that the good or interests of each person must be given equal consideration." Despite this progress on the political theory front, no version of the equal power concept has been formalized technically. How do we formally define "equal weight" of opinions? In this paper, we propose a

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formal definition of equality of power, and show that for quadratic utility functions[1] and a large number of agents[2], there always exists an outcome satisfying this definition.

1.1 Our contribution

Ronald Dworkin's seminal work in private goods economies suggests that each agent should be allowed to impose equal cost on the rest of society [12]. We model this as *externality.* The externality of an agent is the decrease in social welfare (i.e., sum of utilities) for everyone else caused by the existence of that agent. That is, consider the outcome that would be chosen in the absence of that agent, versus the outcome chosen when that agent is included: the externality is the difference in social welfare for the rest of the agents between those two outcomes. We define an agent's power to be her externality, and so equal power requires all agents to have the same externality.

However, we cannot define social welfare in the standard sense, because we do not know the scale of the individual utilities. To define a common scale, we follow the spirit of Dworkin [12] and measure utilities by a metric where the marginal value of additional power for every agent is equal. This is tightly analogous to the definition of equality of resources in terms of equal units of an artificial auction currency, which is exactly the concept of competitive equilibrium from incomes concept from Varian [26]. For additional intuition, imagine that the social planner has a finite amount of power to allocate. In order to maximize social welfare, the marginal value of additional power should be the same for each agent: otherwise, moving power from agents with lower marginal value to agents with higher marginal value would increase the social welfare. We emphasize that the above discussion is not a technical statement, nor is it representative of our actual mathematical model; we include it solely for intuition behind choosing a common scale which equalizes the marginal value of additional power.

Informal statement of results. Our full mathematical model is given in Section 2, but we give an informal description here. We assume agents have quadratic utilities: each agent i has an ideal point $y_i \in \mathbb{R}^m$, along with a weight w_{ij} for each issue j. Agent i's utility for an outcome $x \in \mathbb{R}$ is defined as

$$u_i(x) = -\sum_{j=1}^{m} w_{ij}(y_{ij} - x_j)^2$$

where y_{ij} and x_j are the jth coordinates of y_i and x, respectively. Note that $u_i(x)$ is maximized at $x = y_i$.

To define "marginal value for additional power", we use the following elicitation scheme. Consider an outcome x for the public decision-making problem. We ask each agent to move the outcome towards her ideal point, under the constraint that the externality she imposes on the rest of society is at most some constant γ. When γ is uniform across all agents, this satisfies equality of power.

Our goal, then, is the following. We desire a scaling of utilities $\mathbf{c}$ and an public decision-making outcome x[3] such that all of the following hold:

1. Each agent has equal power. This is achieved by having γ be uniform across agents.
2. Each agent has equal marginal utility for additional power with respect to the elicitation scheme described above (allowing each agent to move the outcome towards their ideal point).
3. The net movement in each direction in the above elicitation scheme is 0.
4. The outcome x maximizes social welfare with respect to $\mathbf{c}$.

For quadratic utilities, we are able to prove existence of such an x and $\mathbf{c}$ in the limit as the number of agents approaches infinity. This leads to Theorem 4.2, whose formal statement comes later. Here $\delta_{ij}(x) \in \mathbb{R}$ represents the amount agent i chooses to move from the current point on issue j. We will often simply denote this by δ_{ij}, and denote the desired shift vector of agent i by δ_i. Note that we do not need to explicitly require that all agents have equal power, as this is ensured by the elicitation scheme (as long as γ is uniform).

[1] See Section 1.1 for a definition.

[2] Specifically, we show that for any finite number of agents, we achieve an approximate version of this equal power outcome, and the approximation error goes to 0 as the number of agents goes to infinity.

[3] This scaling, which we denote $\mathbf{c}$, will be a vector assigning a scaling factor to each agent. The outcome x will be a vector in $\mathbb{R}^m$, where m is the number of issues.

Theorem 4.2 (Informal)**.** *When agent utilities are quadratic, there exists an outcome x and a scaling of agent utilities $\mathbf{c}$ such that as the number of agents goes to infinity, all of the following hold:*

1. *The net movement along each issue (i.e., $\sum_i \delta_{ij}(x)$) is 0.*
2. *Every agent has the same marginal utility for additional power.*
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$.*

The technical statement of the theorem can be found in Section 4. Our proof is quite technically involved. Along with some standard (though involved) Lagrangian duality techniques, we use a fixed point argument whereby we show that a particular infinite-dimensional function admits "almost-fixed" points, i.e., points z where z and $f(z)$ are arbitrarily close (we will end up choosing our scaling $\mathbf{c}$ to be an almost-fixed point of this particular function). Our primary technical contribution is a novel technique for proving existence of approximate fixed points; see Section 4.1 for a more in-depth discussion.

Dependence of marginal utility and social welfare on c. Both the marginal utilities and the social welfare are computed with respect to the utility scale $\mathbf{c}$. The reader may be worried that this makes Theorem 4.2 circular, but it is important to recognize three things. First, the scaling $\mathbf{c}$ is not a free parameter: it is tied down by our requirement that the marginal value for additional power be equal. Second, as mentioned above, this is strongly inspired by the definition of equality of resources in terms of equal amounts of an artificial currency (see Varian [26]).

Third, and most importantly, we argue that it is not meaningful to ask for equal marginal utilities or welfare maximization with respect to the "true" utilities. This is because, in our model, "true" utilities do not really exist: it is not meaningful for ask for the absolute scale of an agent's utility (since there are no monetary payments). The model is not that we are given agents' true utilities and we are scaling them, the model is that we are *defining* a scale of agents' utilities, since some scale is needed in order to maximize social welfare. Inspired by [12], we are choosing a scale that equalizes the marginal utilities.

For some intuition, in the one dimensional case, the outcome specified by Theorem 4.2 turns out to be the median of the agents' ideal points (see Section 3). Furthermore, we argue in Appendix E that our solution concept is not trivial, by showing that an "obvious" choice for $\mathbf{c}$ (specifically, giving each agent the same scaling factor c_i) does not work.

Finally, we briefly discuss incentives and computation. Our query to agents – to provide a desired shift from the current point, under the equal power constraint – is an elicitation method, not a mechanism. Consequently, our result should be thought of only as an existence result. We do not consider mechanism design in a formal sense in this paper, and leave that for future work. We are optimistic about the possibility of an iterative procedure for computing x and $\mathbf{c}$, where on each step, each agent provides a $\delta_i(x)$, and we use $\delta_1 \dots \delta_n$ to compute the next iterate.

1.2 Connections to quadratic voting

It will turn out that our equal power constraint will reduce to a simple quadratic constraint of the form $\sum_j q_j \delta_{ij}^2 \le \gamma$, where each q_j is a positive constant and each j is an issue. Quadratic voting is an increasingly promising voting scheme, both in theory [3, 4, 8, 18, 19, 20, 21, 27] and practice [17, 22, 24]. The fact that our equal power outcome can be implemented with (weighted) quadratic voting leads to a host of promising directions for future work.

In particular, we are optimistic about the possibility of an iterative protocol for computing our desired outcome $(x^*, \mathbf{c})$. As suggested above, consider an iterative algorithm where on each step, we ask each agent for their desired shift $\delta_i(x)$ from the current point x, and use those shifts to compute the next iterate. This algorithm was first studied by Hylland and Zeckhauser in 1979 [18], although instead of the externality constraint (which reduces to $\sum_j q_j \delta_{ij}^2 \le \gamma$), they used subjected each agent to the dimension-symmetric quadratic voting constraint of $\sum_j \delta_{ij}^2 \le \gamma$[4]. They show that their procedure converges to a Pareto optimum.

However, we desire something stronger than just a Pareto optimum. Intuitively, by using a dimension-symmetric constraint, their algorithm ignores the fact that some issues are more important to the population than others. The more the rest of society cares about an issue, the more difficult it should be for an individual

[4] The dimension-symmetric version of this algorithm has also been studied in [3, 4, 8, 17].

to affect the outcome on that issue. This is what Dworkin's quote from the beginning of our paper captures, and what inspires our equal externality constraint. As discussed above, our equal externality constraint will reduce to a constraint of the form $\sum_j q_j \delta_{ij}^2 \leq \gamma$. Each q_j should be interpreted as the aggregate weight society places on issue j.

The distinction between these two constraints is not simply technical. Since each issue is unitless in this model, it not clear what the "right" description of the issue space is, i.e., the right scale for each issue[5]. Our equal externality constraint will be invariant to such rescaling, as intuitively should be the case: if some issue j is rescaled, q_j will simply rescale accordingly. This means that regardless of the representation of the issue space, the outcome described by Theorem 4.2 will be the same. However, Hylland and Zeckhauser's algorithm dimension-symmetric algorithm is extremely vulnerable to this: their outcome will depend dramatically on the precise description of the decision space.

For future work, we are interested in the variant of their algorithm where their dimension-symmetric constraint is replaced with our equal power constraint. This leads to another complication: the right scale for each issue (i.e., q_j) is not known a priori. However, we believe that the right scaling can be discovered as the algorithm progresses based on agents' desired shifts. This is similar to how iterative algorithms for computing private goods market equilibria[6] slowly discover the right prices based on agent demands. All in all, we conjecture that this will lead to an iterative algorithm for public decision-making that both maximizes social welfare, and is consistent with the spirit of equality of resources and envy-freeness studied by economics and philosophy giants such as Ronald Dworkin, Hal Varian, and many more.

Further connections to quadratic voting and second order methods. For the expert reader, we include a brief discussion of some more technical aspects of these connections. Going back to Dworkin [12], he suggests the use of an auction based on equal initial endowments; while he is not explicit about the auction theory involved, he seems to appeal to the idea of a Walrasian auction to which many auction designs converge in large replications of private goods economies with a fixed number of goods [10, 23]. However, the structure of power and quadratic voting is fundamentally different than the linear pricing of a Walrasian auction. For a large population, each agent is only able to suggest a very small shift δ_i from the current point. In particular, the second and higher derivatives of her utility function with respect to δ_i vanish as the number of agents goes to infinity. In order to capture the remaining first derivative, the "pricing" of the δ_i (i.e., the externality constraint on δ_i) should therefore be a quadratic form rather than a linear function, so that the first derivatives of the constraint are linear.

1.3 Other related work

There has been significant recent progress on the theory of public decision-making, some of which with close ties to our work, and some of which using very different approaches. An iterative algorithm which elicits a desired shift from each agent on each step has been studied in [3, 4, 8, 17] and shown to converge under certain assumptions. Furthermore, most of this work does focus specifically on quadratic constraints on the desired shifts. However, none of this work addresses the "weighting" or "rescaling" of dimensions that is crucial to our work (and handled by the q_j constants, as discussed above). For example, [17] focuses on the case where each agent cares about all of the dimensions the same amount.

One can think of the equal power constraint as a pricing mechanism, in the sense that the amount of externality caused (which is equal to $\sum_j q_j \delta_{ij}^2$) is the "price", and each agent has γ units of power to spend. One famous result regarding pricing for public decision-making is that when arbitrary personalized prices are allowed (i.e., the central authority can give agents different prices for the same issue with no restrictions), any Pareto optimal point can be a market equilibrium [14]. A recent paper [16] improved upon this result for the special case of binary issue public decision-making (i.e., each issue has two discrete alternatives, and each agent's ideal point is one of these two alternatives). They gave a reduction by which any binary issue public decision-making problem can be reduced to a private goods market that is "equivalent" in a strong sense, and used this to obtain stronger market equilibrium properties for their special case. Both of these differ from our work in that they use personalized prices to some extent, whereas we subject each agent to

[5] Note that rescaling of the issue space is independent of our scaling $\mathbf{c}$ of the agent utility functions.

[6] Such algorithms are often known as *tâtonnements*.

exactly the same equal power constraint. In this way, our work is arguably more consistent with the spirit of equality of resources.

We briefly mention several non-market approaches. Storable Votes [7] considers a repeated voting context, and permits agents to store their votes for future meetings. In [9], the authors adapt traditional private goods fairness axioms (such as a proportionality) to the public decision-making context for the case where only a discrete set of outcomes are allowed for each issue. The discrete public decision-making problem is also studied by [13], which considers approximate versions of the core, since the (exact) core is not guaranteed to exist in the discrete version of the problem.

The paper proceeds as follows. Section 2 presents the formal model. Section 3 considers the one-dimensional case[7]; this serves as a "warm-up" for the main proof. Since the proof of our main result (Theorem 4.2) is quite involved, we use Section 4 to set up the main result and provide a detailed roadmap of the proof. We then move on to the formal proof. Section 5 contains the fixed point argument that we use to identify our desired outcome x and scaling $\mathbf{c}$. The rest of the proof appears in the appendix. Appendix A provides some additional setup that was omitted from the main body. We then proceed with the rest of the formal proof. Appendix B proves several properties that will be important throughout the proof, such a technical version of the statement "each agent is a small fraction of a large population". Appendix C characterizes each agent's desired shift δ_i, and show that under the choice of x and $\mathbf{c}$ from Section 5, (almost all) the agents have (almost) the same marginal value for additional power. Appendix D handles the last requirement of Theorem 4.2, which is that $\sum_j \delta_{ij}(x)$ is (almost) 0 for each issue j. Also, Appendix E shows that an "obvious" choice of $\mathbf{c}$ (specifically, giving every agent the same scaling factor) is not sufficient for our purposes; this section is solely for intuition.

2 Model

Consider a set of public issues $M = \{1 \dots m\}$. We assume that an outcome for a particular issue is a scalar in $\mathbb{R}$, so an outcome for the overall problem is a vector in $\mathbb{R}^m$. A group of agents need to choose an outcome for this decision-making problem. We assume that each agent is drawn i.i.d. from an integrable probability distribution $p : N \to \mathbb{R}_{\geq 0}$ over possible agent types, where N is the set of agent types. We assume that the distribution is not concentrated too strongly anywhere: specifically, we assume that there exists $p_{max} > 0$ such that $p(i) \leq p_{max}$ for all i. We will use "agents" and "agent types" interchangeably. In general, we will use i and k to refer to agents, and j and ℓ to refer to issues. Since we are holding m fixed and taking n to infinity, we will think of m as a constant (in that we suppress it in asymptotic notation).

Each agent type is specified by an ideal outcome $y_i \in \mathbb{R}^m$ and a weight vector $w_i \in \mathbb{R}^m$. The weight vector represents how much the agent cares about different issues. Let $y_{ij} \in \mathbb{R}$ and $w_{ij} \in \mathbb{R}_{\geq 0}$ denote agent i's ideal outcome and weight for issue j, respectively. Then agent i's utility for an arbitrary point $x \in \mathbb{R}^m$ is

$$u_i(x) = -c_i \sum_{j \in M} w_{ij}(x_j - y_{ij})^2$$

where $c_i \in \mathbb{R}_{>0}$ is the scaling of agent i's utility that we choose. Note that agent i's utility is maximized at $x = y_i$.

Let $\chi \subset \mathbb{R}^m$ be the set of feasible outcomes. We assume that the region of feasible outcomes is bounded and convex. Define d_{max} by

$$d_{max} = \sup_{a,b \in \chi} ||a - b||_2$$

where $||a - b||_2 = \sqrt{\sum_{j \in M}(a_j - b_j)^2}$ is the L_2 norm.

We also assume all agents' weights are bounded above and below. Specifically, we assume there exists $w_{min}, w_{max} > 0$ such that $w_{min} \leq w_{ij} \leq w_{max}$ for all i, j[8]. The assumptions of boundedness of χ and boundedness of w_{ij} for all i, j together imply that the set N is bounded.

[7] Recall that our desired outcome turns out to be the median of the agents' ideal points in this case.

[8] This is really only one assumption, actually: c_i will be invariant to w_i in the sense that if agent i doubles w_i, c_i will halve. This means that only the relative weights matter anyway, so we are essentially assuming that the ratio of each agent's maximum weight dividing by minimum weight is bounded above, i.e., w_i is "well-conditioned".

Equality of power. Let n be the number of agents sampled from p, and let N_s be the random variable representing this set of sampled agents. Rather than focusing on the welfare of the actual set of agents sampled, we use the *expected* societal welfare with respect to the distribution p. However, since the agents are drawn i.i.d., the law of large numbers implies that the two coincide in the limit as $n \to \infty$ anyway. Specifically, for $x \in \chi$, the expected societal welfare $U(x)$ (which depends on the chosen c_i's) is defined by

$$U(x) = \mathbb{E}_{N_s \sim p}\left[\sum_{i \in N_s} u_i(x)\right] = n \int_{i \in N} p(i) u_i(x) \, \mathrm{d}i$$

For our equal power (i.e., equal externality) constraint, we define externality with respect to the expected societal welfare U, not with respect to the welfare of the actual sampled agents N_s. Formally, given a current point $x \in \chi$, we define the externality of a desired shift $\delta \in \mathbb{R}^m$ to be $U(x) - U(x+\delta)$[9]. Thus given a current point x and a small power constant $\gamma > 0$, we present each agent i with the following convex program:

$$\begin{aligned} &\max_{\delta \in \mathbb{R}^m} u_i(x+\delta) \\ &s.t. \quad U(x) - U(x+\delta) \leq \gamma \end{aligned} \tag{1}$$

Let $\delta_i(x) \in \mathbb{R}^m$ be the optimal solution to Program 1 starting from point $x \in \mathbb{R}^m$, and let $\lambda_i(x) \in \mathbb{R}$ be the value of the Lagrange multiplier in the optimal solution. These variables refer only to the optimal solution of agent i's copy of Program 1, not any sort of global optimal solution. Also note that this program implicitly depends on the scaling factors $\mathbf{c}$ (through U).

For a convex program with a differential objective function (such as Program 1), the Lagrange multiplier represents how much we could improve the objective value if the constraint were relaxed[10]. In our case, the objective function here is agent i's utility for outcome $x + \delta$, and the constraint is enforcing that the power used by agent i is at most γ. Thus the Lagrange multiplier $\lambda_i(x)$ is exactly agent i's marginal value for additional power, and this is what we wish to equalize across agents.

Our solution concept. Our solution concept – an *equal-power equal-λ ε-equilibrium* – asks us to chose an outcome $x \in \mathbb{R}^m$ and agent scaling factors $\mathbf{c} \in \mathbb{R}^N_{>0}$ (along with a power constant γ and a particular Lagrange multiplier λ) that satisfies three requirements. First, the expected net movement (the sum of $\delta_i(x)$'s) from the current point is smaller than ε. We use the L_2 norm to express the size of $\sum_i \delta_i(x)$. The second requirement is that all agents except an ε fraction have the same value of $\lambda_i(x)$, up to an ε error. Finally, the selected outcome $x \in \mathbb{R}^m$ must maximize welfare with respect to the chosen agent scaling factors $\mathbf{c} \in \mathbb{R}^N_{>0}$. Since N is a (continuous) distribution over agent types, $\mathbf{c}$ will be an infinite-dimensional vector.

Definition 2.1. *An equal-power equal-λ ε-equilibrium is a outcome $x \in \mathbb{R}^m$, agent scaling factors $\mathbf{c} \in \mathbb{R}^N_{>0}$, power constant γ, and marginal utility $\lambda > 0$ such that*

1. $\mathbb{E}_{N_s \sim p}\left[||\sum_{i \in N_s} \delta_i(x)||_2\right] < \varepsilon$.
2. *The expected number of agents i with $(1-\varepsilon)\lambda \leq \lambda_i(x) \leq \lambda$ is at least $(1-\varepsilon)n$.*
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$, i.e., $x \in \arg\max_{x'} U(x')$.*

We will usually leave x implicit and just write δ_i (which is a vector), δ_{ij} (which is a scalar), and λ_i (which is a scalar). Note that we are only asking for $||\sum_{i \in N_s} \delta_i||_2$ to be small in expectation, but the law of large numbers ensures that the realized value will converge to the expectation with probability 1 as $|N_s| \to \infty$.

Our goal will be to show that for any $\varepsilon > 0$, there exists a large enough n (number of agents) such that an equal-power equal-λ ε-equilibrium exists (for some choice of λ). Specifically, we will choose a fixed x and $\mathbf{c}$ based on the underlying distribution p, agnostic to the set of agents that are actually sampled. We then show that the approximation error goes to 0 as $n \to \infty$.

In the next section, we show that in the one-dimensional case, our desired outcome is the median of the agents' ideal points.

[9] The careful reader may notice that externality is usually defined as the impact on the welfare of everyone *else*, excluding the agent in question. However, since we assume agents to be drawn i.i.d., this distinction is not important.

[10] See Chapter 5 of [6] for an introduction to this type of perturbation analysis.

3 Warm-up: one dimension

We view the one-dimensional case as a warm-up in the sense that the result of this section (Theorem 3.1) will be subsumed by our result for the m-dimensional case (Theorem 4.2). Although the proof of Theorem 4.2 is much more technically involved, the general flow of the proof for the one-dimensional case is similar, so we find it instructive to present first. The main difference is that for the m-dimensional case, the equilibrium point is identified as an approximate fixed point of a particular (somewhat complicated) function. In contrast, for the one-dimensional case, we are able to "guess" that the equilibrium point should be the median. There are several additional small differences, such as the specific bound on λ_i. If the reader is confident and wishes to skip this warm-up, we encourage them to proceed directly to Section 4.

Since we are working with a single dimension, we have $w_i, y_i \in \mathbb{R}$, and agent i's utility function is $u_i(x) = -c_i w_i (x - y_i)^2$. Define x to be the median of the agents' ideal points. Specifically, choose $x \in \mathbb{R}$ such that

$$\int_{i \in N} p(i)\ \text{sgn}(x - y_i)\, \mathrm{d}i = 0$$

That is, the probability of sampling an agent i with $y_i \le x$ is equal to the probability of sampling an agent i with $y_i \ge x$. Since p is continuous, such an x must exist (if there are multiple, choose one arbitrarily).

For each $i \in N$ with $y_i \ne x$, we define c_i to be inversely proportional to her weight w_i and the distance between y_i and x. Agents with $y_i = x$ will turn out to not matter (because this set has measure 0), so we set $c_i = c$ for those agents, where c can be any constant.

$$c_i = \begin{cases} \dfrac{1}{w_i |x - y_i|} & \text{if } y_i \ne x \\ c & \text{if } y_i = x \end{cases}$$

This definition will imply that the outcome is scale-invariant: doubling w_i results in halving c_i, which leads to the same final utility function of $u_i(x) = -\frac{(x-y_i)^2}{|x-y_i|} = -|x - y_i|$. Also, let $q = n \int_{k \in N} p(k) |x - y_k|^{-1}\, \mathrm{d}k$.[11] Since $\int_{k \in N} p(k)|x - y_k|^{-1}\, \mathrm{d}k$ is just some constant (i.e., independent of n), q is $\Theta(n)$.

Theorem 3.1. *For $x, \mathbf{c}$ as defined above, there exists a power constant γ such that the following all hold:*

1. *$|n \int_{i \in N} p(i) \delta_i(x)\, \mathrm{d}i|$ goes to 0 as $n \to \infty$.*
2. *For each agent i except a vanishing fraction[12], λ_i goes to $1/\sqrt{q\gamma}$ as $n \to \infty$.*
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$, i.e., $x \in \arg\max_{x'} U(x')$.*

Note that rather than converging to a specific value, λ_i is approaching $1/\sqrt{q\gamma}$, and q is $\Theta(n)$. However, since we are interested in multiplicative differences in λ_i, this is not a problem. For the m-dimensional case, one product of our more complicated setup will be that λ_i converges to a specific value: specifically, $1/\sqrt{\gamma}$.

Since the point of this section is to give intuition for the main proof (and not to actually prove an interesting result), we are less formal and rigorous than we will be in the proof of the main result. There are also a few (uninformative) parts of the proof that we defer entirely until the proof of the main result.

To start, we consider welfare maximization.

Lemma 3.1. *The outcome x as defined above maximizes welfare with respect to $\mathbf{c}$.*

Proof. Since U is concave and differentiable, and we are maximizing over an unrestricted domain, x maximizes U if and only if derivative of U at x is 0:

$$\frac{d}{dx} U(x) = \frac{d}{dx} n \int_{i \in N} p(i) u_i(x_i)\, \mathrm{d}i = -2 \int_{i \in N} p(i) c_i w_i (x - y_i)\, \mathrm{d}i = -2 \int_{i \in N} p(i) \frac{w_i (x - y_i)}{w_i |x - y_i|}$$

By definition of x, we have $\int_{i \in N} p(i)\ \text{sgn}(x - y_i)\, \mathrm{d}i = \int_{i \in N} p(i) \frac{x - y_i}{|x - y_i|}\, \mathrm{d}i = 0$. Thus $\frac{d}{dx} U(x) = 0$, so $x \in \arg\max_{x'} U(x)$. □

[11] Note that although $|y_i - x|^{-1}$ is undefined at $x = y_i$, its integral is indeed well-defined.

[12] That is, the fraction of agents for whom this does not hold should go to 0 as $n \to \infty$.

Next, we obtain an expression for δ_i in terms of λ_i.

Lemma 3.2. *For each $i \in N$, we have* $\delta_i = \dfrac{(y_i - x)}{|x-y_i|(|x-y_i|^{-1} + \lambda_i q)}$.

Proof. We begin by writing the Lagrangian of Program 1 for an arbitrary agent i:

$$L(\delta_i, \lambda_i) = u_i(x+\delta_i) - \lambda_i\big(U(x) - U(x+\delta_i) - \gamma\big)$$

The KKT conditions imply that the derivative of L with respect to δ_i should be zero for the optimal δ_i:

$$\begin{aligned}
\frac{d}{d\delta_i}L(\delta_i,\lambda_i) &= \frac{d}{d\delta_i}u_i(x+\delta_i) + \lambda_i\frac{d}{d\delta_i}U(x+\delta_i) \\
&= -\frac{2(x+\delta_i-y_i)}{|x-y_i|} - \lambda_i n\int_{k\in N} p(k)\frac{2(x+\delta_i-y_k)}{|x-y_k|}\,\mathrm{d}k \\
&= -\frac{2\delta_i}{|x-y_i|} - \frac{2(x-y_i)}{|x-y_i|} - n\int_{k\in N} p(k)\frac{2\lambda_i\delta_i}{|x-y_k|}\,\mathrm{d}k - 2n\lambda_i\int_{k\in N} p(k)\frac{x-y_k}{|x-y_k|}\,\mathrm{d}k
\end{aligned}$$

By the definition of x, we have $\int_{k\in N} p(k)\,\mathrm{sgn}(x-y_k)\,\mathrm{d}k = \int_{k\in N} p(k)\frac{x-y_k}{|x-y_k|}\,\mathrm{d}k = 0$, so

$$\begin{aligned}
\frac{d}{d\delta_i}L(\delta_i,\lambda_i) &= -\frac{2\delta_i}{|x-y_i|} - \frac{2(x-y_i)}{|x-y_i|} - n\int_{k\in N} p(k)\frac{2\lambda_i\delta_i}{|x-y_k|}\,\mathrm{d}k \\
&= -2\delta_i\Big(|x-y_i|^{-1} + \lambda_i n\int_{k\in N} p(k)|x-y_k|^{-1}\,\mathrm{d}k\Big) - \frac{2(x-y_i)}{|x-y_i|}
\end{aligned}$$

Since $\frac{d}{d\delta_i}L(\delta_i,\lambda_i) = 0$, we get

$$\delta_i = \frac{(y_i-x)}{|x-y_i|\Big(|x-y_i|^{-1} + \lambda_i n\int_{k\in N} p(k)|x-y_k|^{-1}\,\mathrm{d}k\Big)} = \frac{(y_i-x)}{|x-y_i|(|x-y_i|^{-1}+\lambda_i q)}$$

□

We will now use Lemma 3.2 to derive explicit bounds on λ_i. Let $\hat{N} = \{i \in N : |x-y_i| \geq 1/n^{1/4}\}$. Clearly as n goes to ∞, the fraction of agents not in $\hat{N}$ goes to 0. Furthermore, we can always choose the power constant γ to be small enough such that every agent in $\hat{N}$ exhausts her power. Thus for each $i \in \hat{N}$, $U(x) - U(x+\delta_i) = \gamma$.

Lemma 3.3. *For each $i \in \hat{N}$,* $\dfrac{1}{\sqrt{q}}\Big(\dfrac{1}{\sqrt{\gamma}} - \dfrac{1}{\Omega(n^{1/4})}\Big) \leq \lambda_i \leq \dfrac{1}{\sqrt{q\gamma}}$.

Proof. Using arithmetic (we prove this for the more general setting later: see Lemma C.1), $U(x)-U(x+\delta_i) = \delta_i^2 q$. Therefore $\delta_i^2 = \gamma/q$. Also using Lemma 3.2, for each $i \in \hat{N}$ we have

$$\begin{aligned}
\frac{(y_i-x)^2}{(y_i-x)^2(|x-y_i|^{-1}+\lambda_i q)^2} &= \gamma/q \\
|x-y_i|^{-1} + \lambda_i q &= \sqrt{q/\gamma} \\
\lambda_i &= \frac{1}{\sqrt{q\gamma}} - \frac{1}{q|x-y_i|}
\end{aligned}$$

Clearly we have $\lambda_i \leq 1/\sqrt{q\gamma}$. Since $|x-y_i| \geq 1/n^{1/4}$ for $i \in \hat{N}$, we have $\lambda_i \geq \frac{1}{\sqrt{q}}\Big(\frac{1}{\sqrt{\gamma}} - \frac{n^{1/4}}{\sqrt{q}}\Big) = \frac{1}{\sqrt{q}}\Big(\frac{1}{\sqrt{\gamma}} - \frac{1}{\Omega(n^{1/4})}\Big)$. Therefore for each $i \in \hat{N}$, $\frac{1}{\sqrt{q}}\Big(\frac{1}{\sqrt{\gamma}} - \frac{1}{\Omega(n^{1/4})}\Big) \leq \lambda_i \leq \frac{1}{\sqrt{q\gamma}}$. □

Finally, we need the expected net movement to be small: $\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i$ goes to 0 as $n \to \infty$.

Lemma 3.4. *As $n \to \infty$, $|\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i|$ goes to 0.*

The proof for Lemma 3.4 is fairly tedious, even for one dimension. Thus we defer this part of the proof until later, when we formally prove this for the general case.

Lemmas 3.1, 3.3, and 3.4 together imply Theorem 3.1.

4 Main theorem setup

In this section, we state our main theorem (and one variant of the theorem), and provide a roadmap of the proof. Informally, our main result is:

Theorem 4.2 (Informal)**.** *As the number of agents goes to infinity, there exists an outcome x and a scaling of agent utilities $\mathbf{c}$ such that all of the following hold:*

1. *The expected net movement – $||n \int_{i \in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$ – is 0.*
2. *Every agent has the same marginal utility for additional power.*
3. *The outcome x maximizes expected welfare with respect to $\mathbf{c}$.*

This implies that for any $\varepsilon > 0$ and large enough n, there exists an equal-power equal-λ ε-equilibrium.

We state two theorems in this section. The theorem statements refer to a function f that will be defined in Section 5. The variable α is a parameter of f that is used to ensure continuity of f, and will be chosen as a function of n.

Most of the paper is devoted to proving Theorem 4.1, which assumes an exact fixed point of f, and presents approximation bounds on our quantities of interest (i.e, λ_i and $||n \int_{i \in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$) as a function of α and n. An intermediate lemma (Lemma A.1), which appears in Appendix A, shows that there is a choice of α (specifically, $\alpha = n^{-7/8}$) such that the approximation error vanishes as n goes to ∞, also assuming an exact fixed point of f.

In reality, we are not able to prove that f has an exact fixed point. Instead, we show in Section 5 that f has an ε-fixed point for each $\varepsilon > 0$ (where $\varepsilon = 0$ would denote an exact fixed point). Theorem 4.2 states that we can pick ε small enough that using an ε-fixed point of f is good enough.

Theorem 4.1. *Suppose f as defined in Section 5 has an exact fixed point $\mathbf{c}$ for any choice of α and n. Let $x_j = \left(\int_{i \in N} p(i)c_i w_{ij}\,\mathrm{d}i\right)^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i$. Let α be chosen as a function of n so that $\lim_{n \to \infty} n^{3/2}\alpha = \infty$ and $\lim_{n \to \infty} \alpha^{m/2} n^{m/4} = 0$. Then for any ε, there exists α and n such that $(x, \mathbf{c}, \gamma, 1/\sqrt{\gamma})$ is an equal-power equal-λ ε-equilibrium. Specifically:*

1. $||n \int_{i \in N} p(i)\delta_i(x)\,\mathrm{d}i||_2 \leq \dfrac{\gamma + \sqrt{\gamma}}{O(n^2\alpha^2)} + O(\alpha^{m/2} n^{m/4+1})$.
2. *For all i except an expected $O(\alpha^{m/2} n^{m/4})$ fraction, $O\Big(\sqrt{\dfrac{n^{3/2}\alpha}{n^{3/2}\alpha + 1}}\Big)\dfrac{1}{\sqrt{\gamma}} \leq \lambda_i(x) \leq \dfrac{1}{\sqrt{\gamma}}$.*
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$.*

The assumptions of $\lim_{n \to \infty} n^{3/2}\alpha = \infty$ and $\lim_{n \to \infty} \alpha^{m/2} n^{m/4} = 0$ in Theorem 4.1 are necessary for a few parts of the proof to work.

Using Theorem 4.1 and Lemma A.1, we get our final result:

Theorem 4.2. *Let $\mathbf{c}$ be an ε-fixed point of f and let $x_j = \left(\int_{i \in N} p(i)c_i w_{ij}\,\mathrm{d}i\right)^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i$. Let $\alpha = n^{-7/8}$ and $m \geq 6$. Then there exists a small enough ε such that all of the following hold:*

1. $||n \int_{i \in N} p(i)\delta_i(x)\,\mathrm{d}i||_2 \leq (\gamma + \sqrt{\gamma})O(n^{-1/4}) + O(n^{-1/8})$.
2. *For all i except an expected $O(n^{-3/4})$ fraction, $O\Big(\sqrt{\dfrac{n^{5/8}}{n^{5/8} + 1}}\Big)\dfrac{1}{\sqrt{\gamma}} \leq \lambda_i(x) \leq \dfrac{1}{\sqrt{\gamma}}$.*
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$.*

4.1 Proof roadmap

In this section, we state and describe the key lemmas in our proof. The proofs of some of these lemmas involve additional lemmas, but we only include the most important lemmas.

Section 5 is devoted to showing that the function f satisfies the *approximate fixed-point property* (AFPP) [5]: for every $\varepsilon > 0$, f admits an ε-fixed point[13]. In our opinion, this is the most interesting part of the overall proof, and constitutes a novel approach for proving existence of approximate fixed points.

[13] See Section 5 for a formal definition.

The function f will actually be infinite-dimensional, and it turns out (perhaps unsurprisingly) to be challenging to prove fixed point existence for infinite-dimensional functions. Instead, we will analyze a finite-dimensional variant denoted by g_A (A will be any finite set of agents).

We first show that g_A has an exact fixed point for any finite set A (Lemma 5.1). In the following lemma statement, α and $s_1 \dots s_{|A|}$ are parameters of g_A.

Lemma 5.1. *For any* $\alpha \in (0, \frac{\sqrt{n} w_{max}^2 d_{max}^2}{w_{min}}]$*, any finite set* $A \subset N$*, and any nonnegative* $s_1 \dots s_{|A|}$ *that sum to 1, the function* g_A *has a fixed point* $\mathbf{c}^* \in \left[\frac{w_{min}}{w_{max}^2 d_{max}^2}, \frac{\sqrt{n}}{\alpha}\right]^{|A|}$.

We then show that for any $\varepsilon > 0$, there exists an A large enough that any exact fixed point of g can be transformed into an ε-fixed point of f. This part of the argument is quite involved, and uses the following steps: (1) Partition the agent space into arbitrarily small hypercubes. (2) Choose a "representative" from each hypercube in a careful way. (3) Let A be the set of those representatives, let s_ℓ be the measure of p in the ℓth hypercube, and let $\mathbf{c}$ be an exact fixed point of g_A with parameters $s_1 \dots s_{|A|}$. (4) Assign each agent not in A to have the same scaling factor as her representative. (5) Show that for small enough hypercubes, this is an ε-fixed point of f. This results in the following lemma:

Lemma 5.3. *The function* f *satisfies AFPP.*

The rest of the proof appears in the appendix. Also, throughout the rest of the paper after Section 5, we use x and $\mathbf{c}$ to specifically refer to the outcome and agent scaling factors defined in Theorem 4.1, not arbitrary outcomes/agent scaling factors.

In Appendix B, we establish some important properties we will use along the way. First, we show that x as defined in the statement of Theorem 4.1 maximizes welfare with respect to the agent scaling factors $\mathbf{c}$ defined in that theorem statement.

Lemma B.1. *The outcome* x *maximizes societal welfare* $U(x)$ *with respect to* $\mathbf{c}$.

Next, we define $\hat{N}$ as the set of "normal" agents ("normal" will be defined later), and show that the measure of $N \setminus \hat{N}$ is small (i.e., almost all the agents are "normal"). Since we treat m as a constant, the right hand side is $O(\alpha^{m/2} n^{m/4})$; everything else is a constant.

Lemma B.2. *We have* $\displaystyle\int_{i \notin \hat{N}} p(i)\,\mathrm{d}i \leq \alpha^{m/2} n^{m/4} \frac{\pi^{m/2} p_{max}}{\Gamma(\frac{m}{2}+1)} \left(\frac{\sqrt{w_{max}}}{w_{min}}\right)^m$.

Lemma B.4 states that each agent is a small fraction of the overall population, in terms of weight $c_i w_{ij}$ on any issue. Note that the left hand side is an integral over the whole population, and the right hand side is a multiple of $c_i w_{ij}$.

Lemma B.4. *For any agent* $i \in N$, $\displaystyle\int_{k \in N} p(k) c_k w_{kj}\,\mathrm{d}k \geq \frac{w_{min}^2}{4 w_{max}^3 d_{max}^2} \sqrt{n} \alpha c_i w_{ij}$.

Appendix C to devoted to analyzing properties of δ_i and λ_i. First, we show that δ_i obeys a particular expression in terms of λ_i.

Lemma C.2. *For every agent* i *and issue* j, $\delta_{ij} = \dfrac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j}$.

We will define an approximation variable τ_i such that $\lambda_i = \dfrac{1}{\sqrt{\tau_i + \gamma}}$, and show that τ_i is small. This holds for every "normal" agent, i.e., the agents in $\hat{N}$.

Lemma C.5. *For each agent* $i \in \hat{N}$, $\tau_i \leq \dfrac{2\gamma^{3/2} m w_{max}}{n^{3/2} \alpha \beta w_{min} - 2\sqrt{\gamma} m w_{max}}$.

We will also show that $\tau_i > 0$ (this part will be easy). This will allow us to upper- and lower-bound the value of λ_i for each $i \in \hat{N}$. The variable η will be defined later, but we will have $\eta = \Theta(n^{3/2}\alpha)$; by assumption of Theorem 4.1, $\lim_{n\to\infty} n^{3/2}\alpha = \infty$. This means that $\lim_{n\to\infty} \eta = \infty$ as well, and this range of allowable λ_i shrinks to a single point (specifically, $1/\gamma$).

Lemma C.6. *For all* $i \in \hat{N}$, $\sqrt{\frac{\eta}{\eta+1}} \cdot \frac{1}{\sqrt{\gamma}} \le \lambda_i \le \frac{1}{\sqrt{\gamma}}$.

Finally, Appendix D proves an upper bound on $\left|\left|n \int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i\right|\right|_2$: the expected net movement with respect to the current point. Note that $\mathbb{E}_{N_s \sim p}[||\sum_{i\in N_s} \delta_i(x)||_2] = n||\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$.

Lemma D.5. *We have* $\left|\left|n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i\right|\right|_2 \le \frac{\gamma+\sqrt{\gamma}}{\Omega(n^2\alpha^2)} + O(\alpha^{m/2} n^{m/4+1})$.

Lemmas B.1, C.6, and D.5 will together imply Theorem 4.1.

5 The fixed point argument

We will choose our agent scaling factors $\mathbf{c}$ to be an (approximate) fixed point of a particular function f (defined below). This section is devoted to constructing this function and showing that it satisfies the approximate fixed point property (defined in Definition 5.1).

Defining the function f. Let $\mathbb{R}^N_{>0}$ be the set of functions $\mathbf{c} : N \to \mathbb{R}_{>0}$. We can think of each function $\mathbf{c}$ as assigning a scaling factor $c(i) > 0$ to each agent type i. For this section of the paper, we will use the function notation $c(i)$ instead of c_i.

For brevity, for each $j \in M$ define $x_j(\mathbf{c})$ by

$$x_j(\mathbf{c}) = \Big(\int_{i\in N} p(i)c(i)w_{ij}\,\mathrm{d}i\Big)^{-1} \int_{i\in N} p(i)c(i)w_{ij}y_{ij}\,\mathrm{d}i$$

This is a continuous average of all agents' ideal points y_{ij} weighted by $c(i)$ and w_{ij}. We show later that this choice of x maximizes welfare with respect to $\mathbf{c}$ (Lemma B.1).

The function $f : \mathbb{R}^N_{>0} \to \mathbb{R}^N_{>0}$ will take as input a function $\mathbf{c} : N \to \mathbb{R}_{>0}$, and returns a another function $f(\mathbf{c}) : N \to \mathbb{R}_{>0}$. Just as $c(i) \in \mathbb{R}_{>0}$ is the scaling factor that $\mathbf{c}$ assigns to agent i, we use $[f(\mathbf{c})](i) \in \mathbb{R}_{>0}$ to denote the scaling factor that $f(\mathbf{c})$ assigns to agent i. For a small $\alpha > 0$ to be chosen later, we define $[f(\mathbf{c})](i)$ by

$$[f(\mathbf{c})](i) = \frac{\sqrt{n}}{\max\left(\alpha, \sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\int_{k\in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1}}\right)}$$

Intuition behind f. Since we will be choosing $\mathbf{c}$ to be an (approximate) fixed point of f, each c_i will end up approximately equal to $[f(\mathbf{c})](i)$. Consequently, the structure of f gives us a qualitative interpretation of the agent scaling factors $\mathbf{c}$.

Ignore α and $\sqrt{n}$ for now. First, notice that f is invariant to the scale of individual utilities. Specifically, if an agent scales up her weights by a constant factor κ, $[f(\mathbf{c})](i) \approx c(i)$ decreases by a factor of κ: w_{ij}^2 becomes $\kappa^2 w_{ij}^2$, then pull κ out of the square root (still ignoring α). The consequence is that our "common utility scale" defined by $\mathbf{c}$ is invariant to individuals scaling up or down their utility functions, as it should be.

Next, think of $\int_{k\in N} p(k)c(k)w_{kj}\,\mathrm{d}k$ is the aggregate weight placed by society on issue j. Thus each term $w_{ij}^2(y_{ij} - x_j(\mathbf{c}))^2\big(\int_{k\in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1}$ is equal to agent i's weight for issue j (i.e., w_{ij}) times her utility loss on issue j (i.e., $w_{ij}(y_{ij} - x_j(\mathbf{c}))$, as a fraction of society's total weight on that issue. Thus the summation in the denominator can be thought of as expressing how much agent i "disagrees" with the rest of society, with respect to the current point $x(\mathbf{c})$. More intense disagreement leads to a larger denominator, and smaller overall value for $c(i)$.

That said, the real reason for this choice of f is technical: in order for (almost) all agents to have (almost) the same value of λ_i, we will need $c(i)^2$ to be proportional to the expression under the square root for (almost) all agents. This is exactly what a fixed point of f gives us (ignoring α).

Finally, the purposes of $\sqrt{n}$ and α are purely technical. The maximization with α is to ensure that there is no discontinuity in f when the expression under the square root is zero (continuity is required for our fixed point analysis). We will show that α can be chosen so that the properties we desire are not affected. The $\sqrt{n}$ is simply to help certain aspects of the math later on work out smoothly.[14]

Approximate fixed points. Ideally, we would like to show that f has a fixed point. As the reader might expect, showing existence of fixed points in infinite-dimensional spaces can be challenging. Instead, we will show that f admits approximate fixed points:[15]

Definition 5.1. *Let X be a set. We say that a function $f : \mathbb{R}^X_{>0} \to \mathbb{R}^X_{>0}$ satisfies the approximate fixed point property (AFPP) if for every $\varepsilon > 0$, there exists $\mathbf{c}$ such that $|[f(\mathbf{c})](i) - c(i)| < \varepsilon$ for all $i \in X$. We call such a $\mathbf{c}$ an* ε-fixed point.

The rest of this section is devoted to showing that f satisfies AFPP. To do this, we define a function g_A for any finite set of agents A which serves a finite-dimensional approximation of f. We will show that g_A admits an exact fixed point for any set A. To complete the proof, we will show that picking A to be arbitrarily large allows us to approximate f arbitrarily well.

5.1 Showing that f satisfies AFPP

Let $A = \{i_1, i_2 \dots i_{|A|}\}$ be a finite set of agents with nonnegative coefficients $s_1 \dots s_{|A|}$ that sum to 1.With slight abuse of notation, we will use s_{i_k} and s_k interchangeably. We define a function $g_A : \mathbb{R}^{|A|}_{>0} \to \mathbb{R}^{|A|}_{>0}$ by

$$[g_A(\mathbf{c})](i) = \frac{\sqrt{n}}{\max\left(\alpha, \sqrt{\sum_{j \in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \left(\sum_{k \in A} s_k c(k) w_{kj}\right)^{-1}}\right)}$$

for all $i \in A$. That is, g_A takes as input a function $\mathbf{c} : A \to \mathbb{R}_{>0}$ that assigns $c(i)$ to each $i \in A$, and it returns a vector $g_A(\mathbf{c}) \in \mathbb{R}^a_{>0}$ that assigns $[g_A(\mathbf{c})](i)$ each $i \in A$. When $\mathbf{c}$ has a finite domain (such as in the definition of g_A), $x_j(\mathbf{c})$ is defined to be the discrete average of y_i for $i \in A$, weighted by $c(i)$, w_{ij}, and s_i. Formally, $x_j(\mathbf{c}) = (\sum_{i \in A} s_i w_{ij})^{-1} \sum_{i \in A} s_i w_{ij} y_{ij}$. When $\mathbf{c}$ has a continuous domain as in the definition of f, $x_j(\mathbf{c})$ is defined to be the continuous weighted average defined previously.

Lemma 5.1 states that for any set A and any small enough α, g_A has a fixed point. The proof uses Brouwer's fixed point theorem:

Theorem 5.1 (Brouwer's fixed point theorem)**.** *Let ℓ be a positive integer, let $S \subset \mathbb{R}^\ell$ be convex and compact, and let $f : S \to S$ be continuous. Then there exists $\mathbf{c}^* \in S$ such that $f(\mathbf{c}^*) = \mathbf{c}^*$.*

Lemma 5.1. *For any $\alpha \in (0, \frac{\sqrt{n} w_{max}^2 d_{max}^2}{w_{min}}]$, any finite set $A \subset N$, and any nonnegative $s_1 \dots s_{|A|}$ that sum to 1, the function g_A has a fixed point $\mathbf{c}^* \in \left[\frac{w_{min}}{w_{max}^2 d_{max}^2}, \frac{\sqrt{n}}{\alpha}\right]^{|A|}$.*

Proof. Let $S = \left[\frac{w_{min}}{w_{max}^2 d_{max}^2}, \frac{\sqrt{n}}{\alpha}\right]^{|A|}$. Since $c(i) > 0$ for all i for all $\mathbf{c} \in S$, $\sum_{k \in A} s_k c(k) w_{kj}$ is strictly positive. Thus we have $\sum_{j \in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \left(\sum_{k \in A} s_k c(k) w_{kj}\right)^{-1} \geq 0$, so the denominator of $[g_A(\mathbf{c})](i)$ is always real. Furthermore, since $\alpha > 0$, the denominator is always positive, so the function is well-defined and continuous on S. It is also clear that S is convex and compact (and nonempty as long as $\alpha \leq \frac{\sqrt{n} w_{max}^2 d_{max}^2}{w_{min}}$).

[14] It is worth noting that f does depend on n (both explicitly in the numerator, and implicitly through α, which will end up depending on n); this is not a problem, however. Since $x(\mathbf{c})$ is a weighted average of the agents' ideal points, scaling all $c(i)$ by the same amount (which is what $\sqrt{n}$ does) will not affect $x(\mathbf{c})$, the equilibrium outcome. With regards to α, we will need α to go to zero $n \to \infty$ so that the error introduced goes to 0. One can think of this as the impact of α going to zero as $n \to \infty$, so that we achieve an exact equal-power equal-λ equilibrium in the limit.

[15] In general, the approximate fixed point property can be defined for any metric space.

It remains to show that $g_A(\mathbf{c}) \in S$ for all $\mathbf{c} \in S$. First, since the denominator is always at least α, $[g_A(\mathbf{c})](i) \leq \sqrt{n}/\alpha$ for all $\mathbf{c}$ and all i. Next, since $c(i) \geq \frac{w_{min}}{w_{max}^2 d_{max}^2}$ for all $i \in N$ (because $\mathbf{c} \in S$),

$$\begin{aligned}
\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\sum_{k\in A} s_k c(k) w_{kj}\big)^{-1}} &\leq \sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\sum_{k\in A} s_k \frac{w_{min}}{w_{max}^2 d_{max}^2} w_{kj}\big)^{-1}} \\
&= \frac{w_{max} d_{max}}{\sqrt{w_{min}}} \sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\sum_{k\in A} s_k w_{kj}\big)^{-1}} \\
&= \frac{w_{max} d_{max}}{\sqrt{w_{min}}} \sqrt{\sum_{j\in M} w_{max}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\sum_{k\in A} s_k w_{min}\big)^{-1}} \\
&\leq \frac{w_{max}^2 d_{max}}{w_{min}} \sqrt{\sum_{j\in M} (y_{ij} - x_j(\mathbf{c}))^2} \\
&\leq \frac{w_{max}^2 d_{max}^2}{w_{min}}
\end{aligned}$$

Therefore the denominator of $[g_A(\mathbf{c})](i)$ is at most $\frac{w_{max}^2 d_{max}^2}{w_{min}}$, which implies that $[g_A(\mathbf{c})](i) \geq \frac{\sqrt{n} w_{min}}{w_{max}^2 d_{max}^2}$ for all $\mathbf{c} \in S$. Since $n \geq 1$, we have $[g_A(\mathbf{c})](i) \geq \frac{\sqrt{n} w_{min}}{w_{max}^2 d_{max}^2} \geq \frac{w_{min}}{w_{max}^2 d_{max}^2}$, as required.

Therefore $g_A(\mathbf{c}) \in S$ for every $\mathbf{c} \in S$. Thus by Brouwer's fixed point theorem, there exists an $\mathbf{c}^* \in S$ such that $g_A(\mathbf{c}^*) = \mathbf{c}^*$. □

5.2 Using g to approximate f

We next show that for certain choices of A, fixed points of g_A are approximate fixed points of f. The proof approach is as follows:

1. Partition the space of agent types into arbitrarily small hypercubes (Lemma 5.2 shows that this is possible). Thus all agents within a given hypercube will have arbitrarily similar values of y_{ij} and w_{ij}.
2. Choose a "representative" from each hypercube. The representative for the ℓth hypercube will be chosen such that for each $j \in M$, her ideal point y_{ij} and w_{ij} are equal to the weighted average (within the hypercube) of ideal points and weight vectors, respectively. Such an agent is guaranteed to exist within the same hypercube.
3. Let A be the set of representatives, let s_ℓ be the measure of p in the ℓth hypercube, and let $\mathbf{c}_A$ be a fixed point of g_A for coefficients $s_1 \dots s_{|A|}$.
4. Define $\mathbf{c} : N \to \mathbb{R}_{>0}$ so that each agent's scaling factor $c(i)$ is equal to the scaling factor of her representative under $c_A(i)$. Since every agent is in some hypercube, this fully specifies $\mathbf{c}$.
5. Show that $|[f(\mathbf{c})](i) - c(i)|$ is small.

Lemma 5.2. *For some $q > 0$, let $S \subset \mathbb{R}^q$ be a hybercube. Then for any ε, there exists a partition of S into hypercubes $S_1 \dots S_L$ such that for any $\ell \in \{1 \dots L\}$, for all $z, z' \in S_\ell$, $||z - z'||_\infty < \varepsilon$*[16].

Proof. If S is a hypercube, then we can bisect it along every coordinate to create many hypercubes, each with side length half of the original. Starting with N, continue halving the side length in this way until the side length of every hypercube is less than ε. That implies that for any vectors z, z' in the same hypercube, $||z - z'||_\infty < \varepsilon$. □

By assumption, $\chi \subset \mathbb{R}^m$ is bounded, so let $\bar{\chi}$ represent the smallest hypercube that contains χ. Without loss of generality, we can use $\bar{\chi}$ instead, and simply set $p(i) = 0$ for all $y_i \notin \chi$. The set of weight vectors $w_i = (w_{i1} \dots w_{im})$ with $w_{min} \leq w_{ij} \leq w_{max}$ for all i, j is also a hypercube in $\mathbb{R}^m$. Since each agent i is a pair (y_i, w_i), we can write $i \in N \subset \mathbb{R}^{m^2}$, and N too is a hypercube.

Lemma 5.3. *The function f satisfies AFPP.*

[16] We use $||\cdot||_\infty$ to denote the L_∞ norm, which is defined to be the maximum coordinate.

Proof. Recall the definition of AFPP: we need to show that for any $\varepsilon' > 0$, there exists an ε'-fixed point $\mathbf{c}$ of f. Specifically, we need $|[f(\mathbf{c})](i) - c(i)| < \varepsilon'$ for all $i \in N$.

Part 1: Defining the approximate fixed point c. Fix an $\varepsilon > 0$; later on we will choose ε as a function of ε'. Using Lemma 5.2, let $S_1 \dots S_L$ be a partition of N into hypercubes such that for any $\ell \in \{1 \dots L\}$, for all $i, k \in S_\ell$, $||i - k||_\infty < \varepsilon$. This means that for any i, k in the same hypercube and and any $j \in M$, we have

$$|y_{ij} - y_{kj}| < \varepsilon \quad \text{and} \quad |w_{ij} - w_{kj}| < \varepsilon \tag{2}$$

For each $\ell \in \{1 \dots L\}$, let $s_\ell = \int_{i \in S_\ell} p(i)\,\mathrm{d}i$ be the measure of S_ℓ. For each S_ℓ, we will carefully pick a *representative* i_ℓ. For each ℓ, and each j, define $w_{\ell j}^{avg}$ and $y_{\ell j}^{avg}$ by

$$y_{\ell j}^{avg} = \big(\textstyle\int_{i \in S_\ell} p(i) w_{ij}\,\mathrm{d}i\big)^{-1} \int_{i \in S_\ell} p(i) w_{ij} y_{ij}\,\mathrm{d}i \quad \text{and} \quad w_{\ell j}^{avg} = \frac{1}{s_\ell} \int_{i \in S_\ell} p(i) w_{ij}\,\mathrm{d}i$$

Thus for each $j \in M$, $y_{\ell j}^{avg}$ is a weighted average of y_{ij} for $i \in S_\ell$, and $w_{\ell j}^{avg}$ is a weighted average of w_{ij} for $i \in S_\ell$. In particular, $\min_{k \in S_\ell} y_{kj} \le y_{\ell j}^{avg} \le \max_{k \in S_\ell} y_{kj}$, and $\min_{k \in S_\ell} w_{kj} \le w_{\ell j}^{avg} \le \max_{k \in S_\ell} w_{kj}$. Thus since each S_ℓ is a hypercube, each S_ℓ contains an agent i_ℓ with $w_{i_\ell, j} = w_{\ell j}^{avg}$ and $y_{i_\ell j} = y_{\ell j}^{avg}$ for all $j \in M$.

Define $A = \{i_1, i_2 \dots i_L\}$, and let $\mathbf{c}_A$ be a fixed point of g_A with coefficients $s_1 \dots s_\ell$ (which is guaranteed to exist, by Lemma 5.1). Define $\mathbf{c} : N \to \mathbb{R}_{>0}$ so that for each $i \in S_\ell$, $c(i) = c_A(i_\ell)$. In words, we define each agent's scaling factor $c(i)$ to be the same as that of her representative.

Part 2: Properties of c. For any $j \in M$,

$$\begin{aligned}
x_j(\mathbf{c}) &= \big(\textstyle\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1} \int_{k \in N} p(k)c(k)w_{kj}y_{kj}\,\mathrm{d}k && \text{(definition of } x_j(\mathbf{c}) \text{ for continuous } \mathbf{c})\\
&= \big(\textstyle\sum_{\ell=1}^L \int_{k \in S_\ell} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1} \sum_{\ell=1}^L \int_{k \in S_\ell} p(k)c(k)w_{kj}y_{kj}\,\mathrm{d}k && \text{(summing integral over hypercubes)}\\
&= \big(\textstyle\sum_{\ell=1}^L c_A(i_\ell) \int_{k \in S_\ell} p(k)w_{kj}\,\mathrm{d}k\big)^{-1} \sum_{\ell=1}^L c_A(i_\ell) \int_{k \in S_\ell} p(k)w_{kj}y_{kj}\,\mathrm{d}k && \text{(definition of } c(k) \text{ for } k \in S_\ell)\\
&= \big(\textstyle\sum_{\ell=1}^L c_A(i_\ell) \int_{k \in S_\ell} p(k)w_{kj}\,\mathrm{d}k\big)^{-1} \sum_{\ell=1}^L c_A(i_\ell) y_{\ell j}^{avg} \int_{k \in S_\ell} p(k)w_{kj}\,\mathrm{d}k && \text{(definition of } y_{\ell j}^{avg})\\
&= \big(\textstyle\sum_{\ell=1}^L c_A(i_\ell) s_\ell w_{\ell j}^{avg}\big)^{-1} \sum_{\ell=1}^L c_A(i_\ell) y_{\ell j}^{avg} s_\ell w_{\ell j}^{avg} && \text{(definition of } w_{\ell j}^{avg})\\
&= \big(\textstyle\sum_{\ell=1}^L c_A(i_\ell) s_\ell w_{i_\ell, j}\big)^{-1} \sum_{\ell=1}^L c_A(i_\ell) s_\ell w_{i_\ell, j} y_{i_\ell, j} && \text{(definitions of } w_{i_\ell,j} \text{ and } y_{i_\ell,j})\\
&= \big(\textstyle\sum_{k \in A} s_k c_A(k) w_{kj}\big)^{-1} \sum_{k \in A}^L s_k c_A(k) w_{kj} y_{kj} && \text{(definition of } A = \{i_1 \dots i_L\})\\
&= x_j(\mathbf{c}_A) && \text{(definition of } x_j(\mathbf{c}) \text{ for discrete } \mathbf{c})
\end{aligned}$$

Therefore for each $j \in M$, we have $x_j(\mathbf{c}) = x_j(\mathbf{c}_A)$, where the left hand side and right hand side are continuous and discrete weighted averages, respectively.

In the process of the above sequence of equations, we also showed that $\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k = \sum_{k \in A} s_k c_A(k) w_{kj}$. Using this, and the fact that $\mathbf{c}_A$ is a fixed point of g_A, for all $i \in A$ we have

$$\begin{aligned}
c_A(i) &= \sqrt{n}\Big(\max\big(\alpha, \sqrt{\textstyle\sum_{j \in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}_A))^2 \big(\sum_{k \in A} s_k c_A(k) w_{kj}\big)^{-1}}\big)\Big)^{-1}\\
&= \sqrt{n}\Big(\max\big(\alpha, \sqrt{\textstyle\sum_{j \in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1}}\big)\Big)^{-1}\\
&= [f(\mathbf{c})](i)
\end{aligned}$$

Since $c(i) = c_A(i)$ for $i \in A$, we therefore have $c(i) = [f(\mathbf{c})](i)$ exactly when $i \in A$.

Part 3: Showing that $|[f(c)](i) - c(i)|$ is small for every $i \in N$. For $i \in A$, we are done. For $i \notin A$, recall that for each ℓ and all $i \in S_\ell$, $c(i) = c(i_\ell)$ by definition, so $c(i) = c(i_\ell) = [f(\mathbf{c})](i_\ell)$. Therefore it suffices to show that $|[f(\mathbf{c})](i) - [f(\mathbf{c})](i_\ell)|$ is small.

For brevity, let $r(i) = \sum_{j \in M} w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 \big(\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1}$. Then for all $i \in S_\ell$,

$$\begin{aligned}
|r(i) - r(i_\ell)| &= \big(\textstyle\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1} \Big|\sum_{j \in M} \big(w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 - w_{i_\ell, j}^2 (y_{i_\ell j} - x_j(\mathbf{c}))^2\big)\Big| && \text{(defn of } r(i))\\
&\le \big(\textstyle\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1} \sum_{j \in M} \big|w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 - w_{i_\ell, j}^2 (y_{i_\ell j} - x_j(\mathbf{c}))^2\big| && \text{(triangle inequality)}\\
&\le \big(\textstyle\int_{k \in N} p(k)c(k)w_{kj}\,\mathrm{d}k\big)^{-1} \sum_{j \in M} \big|w_{ij}^2 (y_{ij} - x_j(\mathbf{c}))^2 - (w_{ij} + \varepsilon)^2 (y_{ij} - x_j(\mathbf{c}) + \varepsilon)^2\big| && \text{(Eq. 2)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{k\in N} p(k)c(k)w_{kj}\,\mathrm{d}k\right)^{-1}\sum_{j\in M}\left|\varepsilon^4+\varepsilon^2\left(w_{ij}^2(y_{ij}-x_j(\mathbf{c}))^2\right)\right| && \text{(canceling out terms)}\\
&\le \left(\int_{k\in N} p(k)c(k)w_{kj}\,\mathrm{d}k\right)^{-1}\sum_{j\in M}(\varepsilon^4+\varepsilon^2 w_{max}^2 d_{max}^2) && \text{(defn's of } d_{max}, w_{max})\\
&\le \left(\int_{k\in N} p(k)c(k)w_{min}\,\mathrm{d}k\right)^{-1} m(\varepsilon^4+\varepsilon^2 w_{max}^2 d_{max}^2) && \text{(defn of } w_{min})\\
&\le \left(\int_{k\in N} p(k)\frac{w_{min}}{w_{max}^2 d_{max}^2} w_{min}\,\mathrm{d}k\right)^{-1} m(\varepsilon^4+\varepsilon^2 w_{max}^2 d_{max}^2) && \text{(Lemma 5.1)}\\
&\le O(1)\cdot m\cdot(\varepsilon^4+\varepsilon^2\cdot O(1))\\
&= O(\varepsilon^2)
\end{aligned}
$$

Our citation of Lemma 5.1 is because Lemma 5.1 guarantees that every $c(k) \ge \frac{w_{min}}{w_{max}^2 d_{max}^2}$. Thus for each $i \in S_\ell$, $|r(i) - r(i_\ell)| \le O(\varepsilon^2)$.

If $[f(\mathbf{c})](i) = [f(\mathbf{c})](i_\ell) = 1/\alpha$, then $c(i) = [f(\mathbf{c})](i_\ell) = [f(\mathbf{c})](i)$, and we are done. Thus assume at least one does not equal $1/\alpha$. Suppose $f[(\mathbf{c})](i) \ne 1/\alpha$ (the argument is symmetric for the other case), and basic algebra gives us

$$
\begin{aligned}
|[f(\mathbf{c})](i) - [f(\mathbf{c})](i_\ell)| &= \left|\frac{\sqrt{n}}{\sqrt{r(i)}} - \frac{\sqrt{n}}{\max(\alpha, \sqrt{r(i_\ell)})}\right|\\
&\le \sqrt{n}\left|\frac{1}{\sqrt{r(i)}} - \frac{1}{\sqrt{r(i_\ell)}}\right|\\
&= \frac{\sqrt{n}}{\sqrt{r(i)r(i_\ell)}}\left|\sqrt{r(i_\ell)}) - \sqrt{(r(i)}\right|\\
&= \frac{\sqrt{n}}{\sqrt{r(i)r(i_\ell)}(\sqrt{r(i)}+\sqrt{r(i_\ell)})}\,|r(i_\ell) - r(i)|\\
&\le \frac{\sqrt{n}\cdot O(\varepsilon^2)}{\sqrt{r(i)r(i_\ell)}(\sqrt{r(i)}+\sqrt{r(i_\ell)})}
\end{aligned}
$$

Since $r(i), r(i_\ell) \le \alpha$, we have

$$
|[f(\mathbf{c})](i) - [f(\mathbf{c})](i_\ell)| \le \frac{O(\sqrt{n}\varepsilon^2)}{\alpha^3}
$$

and thus $|[f(\mathbf{c})](i) - c(i)| \le O(\sqrt{n}\varepsilon^2)/\alpha^3$. Now, for a fixed ε, taking $n \to \infty$ does make this bound go to infinity. The key here is that ε can be chosen independently of α and n. That is, for any instantiation of f (i.e., for a fixed n and α), we can pick ε to be as small as we want. In particular, since this bound holds uniformly for all $i \in N$ (i.e., with the same ε), for any $\varepsilon' > 0$, there exists ε such that

$$
|[f(\mathbf{c})](i) - c(i)| < \varepsilon'
$$

for all $i \in N$. We conclude that f satisfies AFPP. □

6 Conclusion

In this paper, we proposed and analyzed the concept of *equal power* for multidimensional continuous public decision-making. Drawing fundamental literature in political philosophy and economics, we argued that that equality of power is a natural analog of equality of resources and envy-freeness for public decision-making. Our main result is that for any $\varepsilon > 0$ and a large enough number of agents, an equal-power equal-λ ε-equilibrium is guaranteed to exist; in other words, we achieve an exact equal-power equal-λ equilibrium in the limit. The most interesting part of our proof is the novel fixed point argument presented in Section 5; the rest of our proof appears in the appendix.

There are many possible directions for future work. The first is the possibility of an iterative algorithm for converging to an equal-power equal-λ equilibrium. As discussed in Section 1.2, there is a good reason to be optimistic about the existence of such an algorithm, especially in conjunction with the already extensive quadratic voting literature.

It could also be interesting to extend our results to other utility functions beyond quadratic utilities. A first step could be a "general quadratic utility" of the form $u_i(x) = -(y_i - x)^T W_i (y_i - x)$ for some positive definite matrix W_i. When W_i is a diagonal matrix, this reduces to the form of utility function we studied in this paper. For a general quadratic utility, we conjecture that the equal power constraint would be reduce to a constraint of the form $\delta_i^T Q \delta_i \leq \gamma$. This would correspond to not just a rescaling of the issues, but also a rotation. Our proof does not immediately carry over to general quadratic utility functions, and we suspect that additional mathematical insights are needed.

Possibly the most exciting direction – but also the most ambitious – is extending our results beyond economics that are purely public goods. In general, economies will involve a much richer mix of public goods at different levels of social organization, which are thus partially private when viewed from another resolution (e.g. goods that accrue at the national or city level, but do not spillover beyond these). Efficient, equal budget mechanisms for such societies might offer powerful insights about economic structures that could outperform existing mixtures of capitalism and states.

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A Omitted proofs the main body

Lemma A.1. *Suppose f as defined in Section 5 has an exact fixed point $\mathbf{c}$ for any choice of α and n. Let $x_j = \left(\int_{i\in N} p(i)c_i w_{ij}\,\mathrm{d}i\right)^{-1}\int_{i\in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i$. Let $\alpha = n^{-7/8}$ and $m \geq 6$.*

1. $||n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2 \leq (\gamma + \sqrt{\gamma})O(n^{-1/4}) + O(n^{-1/8})$.
2. *For all i except an expected $O(n^{-3/4})$ fraction,* $O\Big(\sqrt{\frac{n^{5/8}}{n^{5/8}+1}}\Big)\frac{1}{\sqrt{\gamma}} \leq \lambda_i(x) \leq \frac{1}{\sqrt{\gamma}}$.
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$.*

Proof. Assume Theorem 4.1 holds. We need to show the following:

1. $\lim_{n\to\infty}\alpha^{m/2}n^{m/4} = 0$
2. $\lim_{n\to\infty} n^{3/2}\alpha = \infty$
3. $\lim_{n\to\infty}\alpha^{m/2}n^{m/4+1} = 0$
4. $\lim_{n\to\infty}\alpha^2 n^2 = \infty$

The first two points are necessary as conditions of Theorem 4.1, in addition to being necessary for vanishing approximation error. Note that the third point implies the first 1.

Since $m \geq 6$, $O(\alpha^{m/2}n^{m/4+1})$ becomes $O(n^{-3m/16+1}) = O(n^{-18/16+1}) = O(n^{-1/8})$, which does indeed go to zero as $n \to \infty$. This satisfies points 1 and 3.

For the second point, we have $O(n^{3/2}\alpha) = O(n^{3/2-7/8}) = O(n^{5/8})$. Thus $O(n^{3/2}\alpha)$ does go to infinity as $n \to \infty$.

Finally, $\alpha^2 n^2 = n^{2/8} = n^{1/4}$, which goes to infinity as $n \to \infty$. □

Theorem 4.2. *Let $\mathbf{c}$ be an ε-fixed point of f and let $x_j = \left(\int_{i\in N} p(i)c_i w_{ij}\,\mathrm{d}i\right)^{-1}\int_{i\in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i$. Let $\alpha = n^{-7/8}$ and $m \geq 6$. Then there exists a small enough ε such that all of the following hold:*

1. $||n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2 \leq (\gamma + \sqrt{\gamma})O(n^{-1/4}) + O(n^{-1/8})$.
2. *For all i except an expected $O(n^{-3/4})$ fraction,* $O\Big(\sqrt{\frac{n^{5/8}}{n^{5/8}+1}}\Big)\frac{1}{\sqrt{\gamma}} \leq \lambda_i(x) \leq \frac{1}{\sqrt{\gamma}}$.
3. *The outcome x maximizes welfare with respect to $\mathbf{c}$.*

Proof. An ε-fixed point is guaranteed to exist by Lemma 5.3. Assume Lemma A.1 holds. The key here is that ε can be chosen independently of any other parameters (α, n, γ, etc). Furthermore, $\lambda_i(x)$ and $||n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$ will be continuous functions of $\mathbf{c}$. Thus for any ε', there exists ε such that perturbing $\mathbf{c}$ by ε' changes both $||n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$ and each λ_i by at most ε. Since the results are only asymptotic anyway, we can pick ε' small enough that all of the approximations still hold. □

B Important properties to be used later

The rest of the paper is devoted to proving Theorem 4.1. Throughout, we assume that $\mathbf{c}$ is a fixed point of the function f from Section 5, and that $x_j = \left(\int_{i\in N} p(i)c_i w_{ij}\,\mathrm{d}i\right)^{-1}\int_{i\in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i$. Recall that we used the function notation of $c(i)$ only for Section 5; we use the vector notation c_i for the rest of the paper.

B.1 Welfare maximization for quadratic utilities

Lemma B.1. *The outcome x maximizes societal welfare $U(x)$ with respect to* $\mathbf{c}$*.*

Proof. The welfare of an outcome x' is

$$U(x') = n \int_{i\in N} p(i) u_i(x')\,\mathrm{d}i = -n \int_{i\in N} p(i) c_i \sum_{j\in M} w_{ij}(x'_j - y_{ij})^2\,\mathrm{d}i$$

Since U is concave and differentiable, and we are interested in an unconstrained maximization, it suffices to show that the gradient of U at x is 0. Specifically, the partial derivative with respect to x_j should be zero for each j:

$$\begin{aligned}\frac{\partial}{\partial x_j} U(x) =&\; -2n \int_{i\in N} p(i) c_i w_{ij}(x_j - y_{ij})\,\mathrm{d}i \\ =&\; \int_{i\in N} p(i) c_i w_{ij} x_j\,\mathrm{d}i - \int_{i\in N} p(i) c_i w_{ij} y_{ij}\,\mathrm{d}i \\ =&\; x_j \int_{i\in N} p(i) c_i w_{ij}\,\mathrm{d}i - \int_{i\in N} p(i) c_i w_{ij} y_{ij}\,\mathrm{d}i\end{aligned}$$

Substituting in the definition of x_j as given in the statement of Theorem 4.1:

$$\begin{aligned}\frac{\partial}{\partial x_j} U(x) =&\; \big((\textstyle\int_{k\in N} p(k) c_k w_{kj}\,\mathrm{d}k)^{-1} \int_{k\in N} p(k) c_k w_{kj} y_{kj}\,\mathrm{d}k\big) \int_{i\in N} p(i) c_i w_{ij}\,\mathrm{d}i - \int_{i\in N} p(i) c_i w_{ij} y_{ij}\,\mathrm{d}i \\ =&\; \textstyle\int_{i\in N} p(i) c_i w_{ij} y_{ij}\,\mathrm{d}i - \int_{i\in N} p(i) c_i w_{ij} y_{ij}\,\mathrm{d}i \\ =&\; 0\end{aligned}$$

Thus x is indeed optimal for U. □

B.2 The measure of "unusual" agents is small

Let $\hat{N}$ be the set of agents i for whom $c_i = \sqrt{n}/\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) c_k w_{kj}\,\mathrm{d}k\big)^{-1}}$. In ways that will become clear later, $\hat{N}$ is the set of "normal" agents. We need to show that the number of agents not in $\hat{N}$ shrinks to zero as α goes to zero. Here Γ denotes the gamma function.

Lemma B.2. *We have* $\displaystyle\int_{i\notin\hat{N}} p(i)\,\mathrm{d}i \le \alpha^{m/2} n^{m/4} \frac{\pi^{m/2} p_{max}}{\Gamma(\frac{m}{2}+1)} \left(\frac{\sqrt{w_{max}}}{w_{min}}\right)^m$.

Proof. Since $\mathbf{c}$ is a fixed point of f, if $i \notin \hat{N}$, we have $\alpha \ge \sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) c_k w_{kj}\,\mathrm{d}k\big)^{-1}}$. Also, by the definition of f, we have $c_k \le \sqrt{n}/\alpha$ for all $k \in N$. Thus for an arbitrary $i \in N$ we have

$$\begin{aligned}\sqrt{\textstyle\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) c_k w_{kj}\,\mathrm{d}k\big)^{-1}} \ge&\; \sqrt{\textstyle\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) \frac{\sqrt{n}}{\alpha} w_{kj}\,\mathrm{d}k\big)^{-1}} \\ =&\; \frac{\sqrt{\alpha}}{n^{1/4}} \sqrt{\textstyle\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) w_{kj}\,\mathrm{d}k\big)^{-1}} \\ \ge&\; \frac{\sqrt{\alpha}}{n^{1/4}} \sqrt{\textstyle\sum_{j\in M} w_{min}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) w_{max}\,\mathrm{d}k\big)^{-1}} \\ =&\; \frac{\sqrt{\alpha} w_{min}}{n^{1/4}\sqrt{w_{max}}} \sqrt{\textstyle\sum_{j\in M} (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k)\,\mathrm{d}k\big)^{-1}} \\ \ge&\; \frac{\sqrt{\alpha} w_{min}}{n^{1/4}\sqrt{w_{max}}} \sqrt{\textstyle\sum_{j\in M} (y_{ij} - x_j)^2} \\ =&\; \frac{\sqrt{\alpha} w_{min}}{n^{1/4}\sqrt{w_{max}}} ||y_i - x||_2\end{aligned}$$

Thus if $i \notin \hat{N}$,

$$\alpha \geq \frac{\sqrt{\alpha}}{n^{1/4}} ||y_i - x||_2 \frac{w_{min}}{\sqrt{w_{max}}}$$

$$||y_i - x||_2 \leq \frac{\sqrt{\alpha w_{max}} n^{1/4}}{w_{min}}$$

Therefore $i \notin \hat{N}$ only if $||y_i - x||_2 \leq \frac{\sqrt{\alpha w_{max}} n^{1/4}}{w_{min}} w_{min}$. Let B denote the ball with radius $\frac{\sqrt{\alpha w_{max}} n^{1/4}}{w_{min}}$ centered at x. Then we have $\int_{i \notin \hat{N}} p(i)\, \mathrm{d}i \leq \int_{i: y_i \in B} p(i)\, \mathrm{d}i$.

Since $p(i) \leq p_{max}$ for all $i \in N$ by assumption, we have

$$\int_{i: y_i \in B} p(i)\, \mathrm{d}i \leq p_{max} \int_{i: y_i \in B} \mathrm{d}i$$

Since $\int_{i: y_i \in B} \mathrm{d}i$ is just the volume of the m-dimensional unit ball with radius $\frac{\sqrt{\alpha w_{max}} n^{1/4}}{w_{min}}$, we have

$$\int_{i: y_i \in B} \mathrm{d}i = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2} + 1)} \Big(\frac{\sqrt{\alpha w_{max}} n^{1/4}}{w_{min}} \Big)^m$$

□

Since we treat m, p_{max}, w_{min} and w_{max} as constants, we can simply write

$$\int_{i \notin \hat{N}} p(i)\, \mathrm{d}i = O(\alpha^{m/2} n^{m/4})$$

B.3 Each agent is a small fraction of the population

In this section, we show that the weight contribution by any single agent on any issue (i.e., $c_i w_{ij}$) is a small fraction of the weight of the internet population. The main result is Lemma B.4; we first prove Lemma B.3, which lower bounds the aggregate weight of the whole population on issue j.

Lemma B.3. *For all $j \in M$, $\int_{k \in N} p(k) c_k w_{kj}\, \mathrm{d}k \geq \frac{w_{min}^2}{4 w_{max}^2 d_{max}^2} n$.*

Proof. Since $\mathbf{c}$ is assumed to be a fixed point of f, for all $j \in M$ we have

$$\begin{aligned}
\int_{k \in N} p(k) c_k w_{kj}\, \mathrm{d}k &\geq w_{min} \int_{k \in N} p(k) c_k\, \mathrm{d}k \\
&= w_{min} \int_{k \in \hat{N}} \frac{\sqrt{n} p(k)\, \mathrm{d}k}{\max\Big(\alpha, \sqrt{\sum_{\ell \in M} w_{k\ell}^2 (y_{k\ell} - x_\ell)^2 \big(\int_{i \in N} p(i) c_i w_{i\ell}\, \mathrm{d}i \big)^{-1}}\Big)} \\
&\geq w_{min} \sqrt{n} \int_{k \in N} \frac{p(k)\, \mathrm{d}k}{\sqrt{\sum_{\ell \in M} w_{k\ell}^2 (y_{k\ell} - x_\ell)^2 \big(\int_{i \in N} p(i) c_i w_{i\ell}\, \mathrm{d}i \big)^{-1}}} \\
&\geq w_{min} \sqrt{n} \int_{k \in \hat{N}} \frac{p(k)\, \mathrm{d}k}{\sqrt{\sum_{\ell \in M} w_{k\ell}^2 (y_{k\ell} - x_\ell)^2 \big(\int_{i \in N} p(i) c_i w_{min}\, \mathrm{d}i \big)^{-1}}} \\
&\geq w_{min}^{3/2} \sqrt{n} \sqrt{\int_{i \in N} p(i) c_i\, \mathrm{d}i} \int_{k \in \hat{N}} \frac{p(k)\, \mathrm{d}k}{\sqrt{\sum_{\ell \in M} w_{k\ell}^2 (y_{k\ell} - x_\ell)^2}} \\
&\geq \frac{w_{min}^{3/2}}{w_{max}} \sqrt{n} \sqrt{\int_{k \in N} p(k) c_k\, \mathrm{d}k} \int_{k \in \hat{N}} \frac{p(k)\, \mathrm{d}k}{\sqrt{\sum_{\ell \in M} (y_{k\ell} - x_\ell)^2}} \\
&\geq \frac{w_{min}^{3/2}}{w_{max} d_{max}} \sqrt{n} \sqrt{\int_{k \in N} p(k) c_k\, \mathrm{d}k} \int_{k \in \hat{N}} p(k)\, \mathrm{d}k
\end{aligned}$$

By Lemma B.2, $\int_{k \notin \hat{N}} p(k)\, \mathrm{d}k = O(\alpha^{m/2} n^{m/4})$, and we know that $\lim_{n \to \infty} \alpha^{m/2} n^{m/4} = 0$ by assumption of Theorem 4.1. Thus for large enough n, $\int_{k \in \hat{N}} p(k)\, \mathrm{d}k \geq 1/2$, so

$$\begin{aligned} w_{min} \int_{k \in N} p(k) c_k \, \mathrm{d}k &\geq \frac{w_{min}^{3/2}}{w_{max} d_{max}} \sqrt{n} \sqrt{\int_{k \in N} p(k) c_k \, \mathrm{d}k} \int_{k \in \hat{N}} p(k)\, \mathrm{d}k \\ &\geq \frac{w_{min}^{3/2}}{2 w_{max} d_{max}} \sqrt{n} \sqrt{\int_{k \in N} p(k) c_k \, \mathrm{d}k} \end{aligned}$$

Therefore $w_{min} \int_{k \in N} p(k) c_k \, \mathrm{d}k \geq \frac{w_{min}^{3/2}}{2 w_{max} d_{max}} \sqrt{n} \sqrt{\int_{k \in N} p(k) c_k \, \mathrm{d}k}$, so

$$\begin{aligned} \sqrt{\int_{k \in N} p(k) c_k \, \mathrm{d}k} &\geq \frac{w_{min}^{1/2}}{2 w_{max} d_{max}} \sqrt{n} \\ \int_{k \in N} p(k) c_k \, \mathrm{d}k &\geq \frac{w_{min}}{4 w_{max}^2 d_{max}^2} n \end{aligned}$$

Therefore

$$\int_{k \in N} p(k) c_k w_{kj} \, \mathrm{d}k \geq \frac{w_{min}^2}{4 w_{max}^2 d_{max}^2} n$$

as required.

□

Lemma B.4. *For any agent $i \in N$, $\displaystyle\int_{k \in N} p(k) c_k w_{kj} \, \mathrm{d}k \geq \frac{w_{min}^2}{4 w_{max}^3 d_{max}^2} \sqrt{n} \alpha c_i w_{ij}$.*

Proof. Since $c_i \leq \sqrt{n}/\alpha$ for all $i \in N$ and $w_{ij} \leq w_{max}$, the right hand side is at most $\frac{w_{min}^2}{4 w_{max}^2 d_{max}^2} n$. Applying Lemma B.3 completes the proof. □

For brevity, let $\beta = \frac{w_{min}^2}{4 w_{max}^3 d_{max}^2}$. Thus $\int_{k \in N} p(k) c_k w_{kj} \, \mathrm{d}k \geq \beta \sqrt{n} \alpha c_i w_{ij}$.

C Characterizing $\boldsymbol{\delta_i}$ and $\boldsymbol{\lambda_i}$

In this section, we derive an expression for δ_i in terms of λ_i, then provide almost tight upper and lower bounds on λ_i. Lemma C.2 gives the expression for δ_i in terms of λ_i, and Lemma C.6 gives the upper and lower bounds for λ_i.

C.1 Deriving an expression for $\boldsymbol{\delta_i}$ in terms of $\boldsymbol{\lambda_i}$

First, we show that the equal power constraint of Program 1 can be reduced to a simpler form.

Lemma C.1. *Then the equal power constraint is equivalent to*

$$\sum_{j \in M} \delta_{ij}^2 \big(n \int_{k \in N} p(k) c_k w_{kj} \, \mathrm{d}k \big) \leq \gamma$$

Proof. For quadratic utilities, we can rewrite $U(x + \delta_i)$ as

$$U(x + \delta_i) = U(x) + \delta_i^T (\nabla_x U)(x) + \frac{1}{2} \delta_i^T (\nabla_x^2 U)(x) \delta_i$$

where the gradient is just with respect to x. For brevity, we will omit the parentheses and just write $\nabla_x U(x)$ etc. By Lemma B.1, x maximizes U with respect to $\mathbf{c}$. Therefore $\nabla_x U(x) = 0$, so

$$U(x) - U(x + \delta_i) = U(x) - \big(U(x) + \delta_i^T \nabla_x U(x) + \frac{1}{2} \delta_i^T \nabla_x^2 U(x) \delta_i \big)$$

$$= -\frac{1}{2}\delta_i^T \nabla_x^2 U(x)\delta_i$$

This makes the constraint of equal power reduce to

$$-\frac{1}{2}\delta_i^T \nabla_x^2 U(x)\delta_i \leq \gamma$$

Next, recall that $U(x) = n \int_{i\in N} p(i)u_i(x)\,\mathrm{d}i$ by definition. Therefore

$$\frac{\partial}{\partial x_j}U(x) = \frac{\partial}{\partial x_j}\Big(- n \int_{k\in N} p(k)\big(c_k \sum_{j\in M} w_{kj}(x_j - y_{kj})^2\big)\,\mathrm{d}k\Big) = -2n \int_{k\in N} p(k)c_k w_{kj}(x_j - y_{kj})\,\mathrm{d}k$$

which means that $\frac{\partial^2}{\partial x_j \partial x_\ell} = 0$ whenever $j \neq \ell$, for $j = \ell$ we have

$$\frac{\partial^2}{\partial x_j^2}U(x) = -2n \int_{k\in N} p(k)c_k w_{kj}\,\mathrm{d}k$$

Thus $\nabla_x^2 U(x)$ is a diagonal matrix with entry $n \int_{k\in N} p(k)c_k w_{kj}\,\mathrm{d}k$ in the jth row, so the equal power constraint simplifies to $\sum_{j\in M} \delta_{ij}^2 \big(n \int_{k\in N} p(k)c_k w_{kj}\,\mathrm{d}k\big) \leq \gamma$, as required. □

For brevity, define q_j by

$$q_j = n \int_{k\in N} p(k)c_k w_{kj}\,\mathrm{d}k$$

Thus the equal power constraint is equivalent to $\sum_{j\in M} q_j\delta_{ij}^2 \leq \gamma$. We can think of q_j as how much the population in aggregate cares about issue j. The expression $\sum_{j\in M} q_j\delta_{ij}^2$ indicates that the more the population cares about issue j, the more power it required to move on that issue.

Lemma C.2. *For every agent i and issue j, $\delta_{ij} = \frac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j}$.*

Proof. Define the Lagrangian by

$$\begin{aligned} L(\delta_i, \lambda_i) &= u_i(x + \delta_i) - \lambda_i\big(U(x) - U(x + \delta_i) - \gamma\big) \\ &= - c_i \sum_{j\in M} w_{ij}(x + \delta_i - y_{ij})^2 - \lambda_i\big(\sum_{j\in M} q_j\delta_{ij}^2 - \gamma\big) \end{aligned}$$

where the second line used Lemma C.1.

For $\delta_i = \mathbf{0}$, we have $U(x) - U(x + \delta_i) = 0 < \gamma$, so we have strong duality by Slater's condition. Since the convex is convex, the optimal solution must satisfy the KKT conditions; in particular, stationarity:

$$\frac{\partial}{\partial \delta_{ij}}L(\delta_i, \lambda_i) = 0$$

for all $j \in M$. Therefore for all j,

$$\begin{aligned} -2c_i w_{ij}(x_j + \delta_{ij} - y_{ij}) - 2\lambda_i q_j\delta_{ij} &= 0 \\ (c_i w_{ij} + \lambda_i q_j)\delta_{ij} + c_i w_{ij}(x_j - y_{ij}) &= 0 \\ \delta_{ij} &= \frac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j} \end{aligned}$$

as required. □

C.2 Bounding $\boldsymbol{\lambda_i}$

For each agent i, we have $m+1$ unknowns: $\delta_{i1} \ldots \delta_{im}$, and λ_i. The previous section gave us m equations: one for each δ_{ij}. For our last equation, we show that we can pick γ small enough such that for most of the agents, the equal power constraint holds with equality. Specifically, the power constraint will hold with equality for every agent in $\hat{N}$ (and we know $N \setminus \hat{N}$ to have small measure by Lemma B.2).

Lemma C.3. *There exists a γ such that the power constraint is tight for all $i \in \hat{N}$, i.e., $\sum_{j\in M} q_j \delta_{ij}^2 = \gamma$.*

Proof. For all agents $i \in \hat{N}$, we know that

$$\begin{aligned}
\alpha &\leq \sqrt{\textstyle\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) c_k w_{kj} \, \mathrm{d}k \big)^{-1}} \\
&\leq \sqrt{\textstyle\sum_{j\in M} w_{max}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) \frac{w_{min}}{w_{max}^2 d_{max}^2} w_{min} \, \mathrm{d}k \big)^{-1}} \\
&\leq \frac{w_{max}^2 d_{max}}{w_{min}} \sqrt{\textstyle\sum_{j\in M} (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) \, \mathrm{d}k \big)^{-1}} \\
&\leq \frac{w_{max}^2 d_{max}}{w_{min}} ||y_i - x||_2
\end{aligned}$$

Thus $||y_i - x||_2 \geq \frac{\alpha w_{min}}{w_{max}^2 d_{max}}$. Suppose for sake of contradiction that for all $\gamma > 0$, there is an agent $i \in \hat{N}$ whose power constraint is not tight. That would imply that there are agents in $\hat{N}$ arbitrarily close to x, which we just showed is not true. We conclude that there exists a $\gamma > 0$ such that $\sum_{j\in M} q_j \delta_{ij}^2 = \gamma$ for all $i \in \hat{N}$. □

Note that if Lemma C.3 holds for some $\gamma > 0$, it also holds for any $\gamma' \in (0, \gamma]$. In particular, later on we will require that $\gamma \leq 1$.

Recall that $\hat{N}$ is the set of agents i for whom $c_i = \sqrt{n}/\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(\int_{k\in N} p(k) c_k w_{kj} \, \mathrm{d}k \big)^{-1}}$. This implies $c_i = 1/\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \big(n \int_{k\in N} p(k) c_k w_{kj} \, \mathrm{d}k \big)^{-1}} = 1/\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1}}$. For each $i \in \hat{N}$, define τ_i by

$$\tau_i = \frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} - \sum_{j\in M} q_j \delta_{ij}^2$$

By Lemma C.3,

$$\frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} = \tau_i + \gamma$$

We know that $c_i = 1/\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1}}$ for every $i \in \hat{N}$, so $\frac{1}{\lambda_i^2} = \tau_i + \gamma$, and therefore

$$\lambda_i = \frac{1}{\sqrt{\tau_i + \gamma}}$$

C.3 Bounding τ_i

Our goal is to show that for all $i \in \hat{N}$, λ_i is close to $1/\sqrt{\gamma}$. To do this, we need to show that τ_i is small and nonnegative. Nonnegativity is (much) easier, so we begin with that.

Lemma C.4. *For all $i \in \hat{N}$, $\tau_i > 0$.*

Proof. Plugging in the expression for δ_{ij} from Lemma C.2, we have

$$\begin{aligned}
\sum_{j\in M} q_j \delta_{ij}^2 &= \sum_{j\in M} \frac{q_j c_i^2 w_{ij}^2 (y_{ij} - x_j)^2}{(c_i w_{ij} + \lambda_i q_j)^2} \\
&< \sum_{j\in M} \frac{q_j c_i^2 w_{ij}^2 (y_{ij} - x_j)^2}{(\lambda_i q_j)^2} \\
&= \frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1}
\end{aligned}$$

Therefore

$$\tau_i > \frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} - \frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} = 0$$

□

Next, we prove an upper bound on τ_i. The upper bound expression in Lemma C.5 appears quite complicated, but notice that the denominator contains $n^{3/2}\alpha$, and we know that $\lim_{n\to\infty} n^{3/2}\alpha = \infty$.

Lemma C.5. *For each agent $i \in \hat{N}$, $\tau_i \le \frac{2\gamma^{3/2} m w_{max}}{n^{3/2}\alpha\beta w_{min} - 2\sqrt{\gamma} m w_{max}}$.*

Proof. We begin the proof with some algebraic manipulations on the definition of τ_i:

$$\begin{aligned}
\tau_i &= \frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} - \sum_{j\in M} q_j \delta_{ij}^2 \\
&= \frac{c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} - \sum_{j\in M} q_j \Big(\frac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j}\Big)^2 \\
&= c_i^2 \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \Big(\frac{1}{\lambda_i^2 q_j} - \frac{q_j}{(c_i w_{ij} + \lambda_i q_j)^2}\Big) \\
&= c_i^2 \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \frac{(c_i w_{ij} + \lambda_i q_j)^2 - \lambda_i^2 q_j^2}{\lambda_i^2 q_j (c_i w_{ij} + \lambda_i q_j)^2} \\
&= c_i^2 \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \frac{c_i^2 w_{ij}^2 + \lambda_i^2 q_j^2 + 2 c_i w_{ij} \lambda_i q_j - \lambda_i^2 q_j^2}{\lambda_i^2 q_j (c_i w_{ij} + \lambda_i q_j)^2} \\
&= c_i^2 \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 \frac{c_i^2 w_{ij}^2 + 2 c_i w_{ij} \lambda_i q_j}{\lambda_i^2 q_j (c_i w_{ij} + \lambda_i q_j)^2} \\
&= \frac{c_i^3}{\lambda_i^2} \sum_{j\in M} w_{ij}^3 (y_{ij} - x_j)^2 \frac{c_i w_{ij} + 2\lambda_i q_j}{q_j (c_i w_{ij} + \lambda_i q_j)^2} \\
&= \frac{2c_i^3}{\lambda_i^2} \sum_{j\in M} w_{ij}^3 (y_{ij} - x_j)^2 \frac{c_i w_{ij} + 2\lambda_i q_j}{q_j (c_i w_{ij} + \lambda_i q_j)(2 c_i w_{ij} + 2\lambda_i q_j)} \\
&\le \frac{2c_i^3}{\lambda_i^2} \sum_{j\in M} w_{ij}^3 (y_{ij} - x_j)^2 \frac{1}{q_j (c_i w_{ij} + \lambda_i q_j)}
\end{aligned}$$

By Lemma C.1 (and the definition of q_j), each agent's power constraint is $\sum_{j\in M} \delta_{ij}^2 q_j \le \gamma$. This means that for all $i \in N$ and $j \in M$, $\sqrt{\delta_{ij}} \le \sqrt{\gamma} q_j^{-1/2}$. Also recall that $\delta_{ij} = \frac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j}$ (Lemma C.2). Therefore

$$\begin{aligned}
\tau_i &\le \frac{2c_i^3}{\lambda_i^2} \sum_{j\in M} w_{ij}^3 (y_{ij} - x_j)^2 \frac{1}{q_j (c_i w_{ij} + \lambda_i q_j)} \\
&= \frac{2c_i^2}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 (y_{ij} - x_j) \delta_{ij} q_j^{-1} \\
&\le \frac{2c_i^2 \sqrt{\gamma}}{\lambda_i^2} \sum_{j\in M} w_{ij}^2 |y_{ij} - x_j| q_j^{-1/2} q_j^{-1}
\end{aligned}$$

Next, let $\Delta_i = \max_{j\in M} |y_{ij} - x_j| q_j^{-1/2}$. Recall that $c_i = 1/\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1}}$ for every $i \in \hat{N}$. Since $w_{ij} \ge w_{min}$, we have

$$\sqrt{\sum_{j\in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1}} \ge w_{min} \sqrt{\sum_{j\in M} (y_{ij} - x_j)^2 q_j^{-1}} \ge w_{min} \Delta_i$$

Thus $c_i \le \frac{1}{w_{min}\Delta_i}$ for all $i \in \hat{N}$. Therefore

$$\tau_i \le \frac{2c_i\sqrt{\gamma}}{w_{min}\Delta_i\lambda_i^2}\sum_{j\in M} w_{ij}^2\Delta_i q_j^{-1} = \frac{2c_i\sqrt{\gamma}}{w_{min}\lambda_i^2}\sum_{j\in M} w_{ij}^2 q_j^{-1}$$

Lemma B.4 implies that for all $j \in M$, $\int_{k\in N} p(k)c_k w_{kj}\,\mathrm{d}k \ge \beta\sqrt{n}\alpha c_i w_{ij}$. Using the definition of q_j, we get $q_j \ge \beta n^{3/2}\alpha c_i w_{ij}$, so $c_i w_{ij} q_j^{-1} \le \frac{1}{\beta n^{3/2}\alpha}$. Therefore

$$\begin{aligned}
\tau_i &\le \frac{2\sqrt{\gamma}}{w_{min}\lambda_i^2}\sum_{j\in M} w_{ij}(c_i w_{ij} q_j^{-1}) \\
&\le \frac{2\sqrt{\gamma}}{\beta w_{min} n^{3/2}\alpha\lambda_i^2}\sum_{j\in M} w_{ij} \\
&\le \frac{2\sqrt{\gamma} m w_{max}}{\beta w_{min} n^{3/2}\alpha\lambda_i^2} \\
&= (\tau_i+\gamma)\frac{2\sqrt{\gamma} m w_{max}}{\beta w_{min} n^{3/2}\alpha}
\end{aligned}$$

We can now solve for τ_i:

$$\begin{aligned}
\tau_i n^{3/2}\alpha\beta w_{min} &\le 2(\tau_i+\gamma)\sqrt{\gamma} m w_{max} \\
\tau_i(n^{3/2}\alpha\beta w_{min} - 2\sqrt{\gamma} m w_{max}) &\le 2\gamma^{3/2} m w_{max} \\
\tau_i &\le \frac{2\gamma^{3/2} m w_{max}}{n^{3/2}\alpha\beta w_{min} - 2\sqrt{\gamma} m w_{max}}
\end{aligned}$$

Note that we assumed $\beta w_{min} n^{3/2}\alpha - 2\sqrt{\gamma} m w_{max} > 0$ when dividing both sides by that quantity. This is true as $n \to \infty$, since $\lim_{n\to\infty} n^{3/2}\alpha = \infty$ by assumption. □

For brevity, let $\eta = \frac{n^{3/2}\alpha\beta w_{min} - 2\sqrt{\gamma} m w_{max}}{2 m w_{max}}$. Thus $\tau \le \gamma^{3/2}/\eta$. Since we can always pick $\gamma \le 1$, we have $\tau \le \gamma/\eta$. Recall that β is a constant, so $\eta = \Theta(n^{3/2}\alpha)$.

Lemma C.6. *For all* $i \in \hat{N}$, $\sqrt{\frac{\eta}{\eta+1}} \cdot \frac{1}{\sqrt{\gamma}} \le \lambda_i \le \frac{1}{\sqrt{\gamma}}$.

Proof. By Lemma C.5,

$$\begin{aligned}
\lambda_i &= \frac{1}{\sqrt{\tau_i+\gamma}} \\
&\ge \frac{1}{\sqrt{\gamma/\eta+\gamma}} \\
&\ge \frac{\sqrt{\eta}}{\sqrt{\gamma+\gamma\eta}} \\
&= \sqrt{\frac{\eta}{\eta+1}} \cdot \frac{1}{\sqrt{\gamma}}
\end{aligned}$$

This satisfies the first inequality. The second follows immediately from the fact that $\tau_i \ge 0$. □

D Bounding the net movement: $||\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$

Finally, we need to show that $||\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i||_2$ is small. Recall that by Lemma C.2,

$$\delta_i = \frac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j}$$

for all $i \in N$. We start by defining two approximations to δ_i:

$$\delta_i' = \frac{c_i w_{ij}(y_{ij} - x_j)}{\lambda_i q_j}$$
$$\delta_i'' = c_i w_{ij}(y_{ij} - x_j) q_j^{-1} \sqrt{\gamma}$$

We first show that $\big|\big|\int_{i \in N} p(i)\delta_i''(x)\,\mathrm{d}i\big|\big|_2 = 0$ exactly. Next, we show that δ_i'' approximates δ_i' well, and that δ_i' approximates δ_i well, for each $i \in \hat{N}$. Finally, we show that the agents not in $\hat{N}$ do not matter too much, since their combined measure is small (Lemma B.2).

Lemma D.1.

$$\big|\big|\int_{i \in N} p(i)\delta_i''(x)\,\mathrm{d}i\big|\big|_2 = 0$$

Proof.

$$\begin{aligned}\big|\big|\int_{i \in N} p(i)\delta_i''(x)\,\mathrm{d}i\big|\big|_2 &= \big|\big|\int_{i \in N} p(i)c_i w_{ij}(y_{ij} - x_j) q_j^{-1}\sqrt{\gamma}\,\mathrm{d}i\big|\big|_2 \\ &= \big|\big|\sqrt{\gamma} q_j^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i - \sqrt{\gamma} q_j^{-1} \int_{i \in N} p(i) c_i w_{ij} x_j\,\mathrm{d}i\big|\big|_2 \\ &= \Big|\Big|\sqrt{\gamma} q_j^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i - \sqrt{\gamma} q_j^{-1} \Big(\int_{i \in N} p(i) c_i w_{ij}\,\mathrm{d}i\Big) x_j\Big|\Big|_2\end{aligned}$$

Since $x_j = \big(\int_{i \in N} p(i)c_i w_{ij}\,\mathrm{d}i\big)^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i$, we have

$$\begin{aligned}\big|\big|\int_{i \in N} p(i)\delta_i''(x)\,\mathrm{d}i\big|\big|_2 &= \big|\big|\sqrt{\gamma} q_j^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i - \sqrt{\gamma} q_j^{-1} \int_{i \in N} p(i)c_i w_{ij} y_{ij}\,\mathrm{d}i\big|\big|_2 \\ &= 0\end{aligned}$$

as required.

□

Lemma D.2. *For each $i \in \hat{N}$,*

$$||\delta_i' - \delta_i||_2 \le \frac{2\gamma d_{max}}{\beta^2 n^3 \alpha^2}$$

Proof. The expression $\delta_i' - \delta_i$ reduces to:

$$\begin{aligned}\frac{c_i w_{ij}(y_{ij} - x)}{\lambda_i q_j} - \frac{c_i w_{ij}(y_{ij} - x)}{c_i w_{ij} + \lambda_i q_j} &= c_i w_{ij}(y_{ij} - x_j)\frac{c_i w_{ij} + \lambda_i q_j - \lambda_i q_j}{\lambda_i q_j (c_i w_{ij} + \lambda_i q_j)} \\ &= \frac{c_i^2 w_{ij}^2 (y_{ij} - x_j)}{\lambda_i q_j (c_i w_{ij} + \lambda_i q_j)}\end{aligned}$$

Then using Lemmas B.4 and C.6, we have

$$\begin{aligned}||\delta_i' - \delta_i||_2^2 &= \frac{c_i^4 w_{ij}^4 (y_{ij} - x_j)^2}{\lambda_i^2 q_j^2 (c_i w_{ij} + \lambda_i q_j)^2} \\ &\le \frac{c_i^4 w_{ij}^4 (y_{ij} - x_j)^2}{\lambda_i^4 q_j^4} \\ &\le \frac{d_{max}^2}{\lambda_i^4} \cdot \Big(\frac{c_i w_{ij}}{q_j}\Big)^4 \\ &\le \frac{d_{max}^2}{\lambda_i^4 \beta^4 n^6 \alpha^4} \\ &\le \frac{\gamma^2}{\beta^4 n^6 \alpha^4}\Big(\frac{\eta + 1}{\eta}\Big)^2 d_{max}^2\end{aligned}$$

As long as $\eta \geq 1$ (which is of course true as n approaches infinity), we have

$$||\delta_i' - \delta_i||_2^2 \leq 4\frac{\gamma^2}{\beta^4 n^6 \alpha^4} d_{max}^2$$

Altogether, this implies that

$$||\delta_i' - \delta_i||_2 \leq \frac{2\gamma d_{max}}{\beta^2 n^3 \alpha^2}$$

as required. □

Lemma D.3. *For all* $i \in \hat{N}$,

$$||\delta_i' - \delta_i''||_2 \leq \frac{\sqrt{\gamma} d_{max}}{\beta \eta n^{3/2} \alpha}$$

Proof. We have

$$\begin{aligned}
\frac{1}{\lambda_i} - \sqrt{\gamma} &\leq \sqrt{\gamma}\Big(\sqrt{\frac{\eta+1}{\eta}} - 1\Big) \\
&= \sqrt{\gamma}\Big(\sqrt{\frac{\eta+1}{\eta}} - \sqrt{\frac{\eta}{\eta}}\Big) \\
&= \sqrt{\gamma}\Big(\sqrt{\frac{\eta+1}{\eta}} - \sqrt{\frac{\eta}{\eta}}\Big) \\
&= \sqrt{\gamma}\frac{\sqrt{\eta+1} - \sqrt{\eta}}{\sqrt{\eta}} \\
&= \sqrt{\gamma}\frac{\eta+1-\eta}{\sqrt{\eta}(\sqrt{\eta+1} + \sqrt{\eta})} \\
&\leq \frac{\sqrt{\gamma}}{\eta}
\end{aligned}$$

Thus $||\delta_i' - \delta_i''||_2$ is bounded by

$$\begin{aligned}
||\delta_i' - \delta_i''||_2 &\leq \Big|\Big|\frac{c_i w_{ij}(y_{ij} - x)}{\lambda_i q_j} - c_i\sqrt{\gamma} q_j^{-1} w_{ij}(y_{ij} - x)\Big|\Big|_2 \\
&\leq \Big|\Big|\frac{\sqrt{\gamma}}{\eta} c_i q_j^{-1} w_{ij}(y_{ij} - x)\Big|\Big|_2 \\
&\leq \frac{\sqrt{\gamma}}{\eta}||c_i q_j^{-1} w_{ij}(y_{ij} - x)||_2 \\
&= \frac{\sqrt{\gamma}}{\eta}\sqrt{(y_{ij} - x_j)^2 (c_i w_{ij} q_j^{-1})^2} \\
&= \frac{\sqrt{\gamma}}{\eta}||y_i - x||_2 c_i w_{ij} q_j^{-1} \\
&\leq \frac{\sqrt{\gamma} d_{max}}{\beta \eta n^{3/2} \alpha}
\end{aligned}$$

□

Lemma D.4.

$$\Big|\Big|\int_{i \notin \hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\Big|\Big|_2 = O(\alpha^{m/2} n^{m/4})$$

Proof. For δ_i, we trivially have $||\delta_i||_2 \leq d_{max}$. For δ_i'', by Lemma B.4 (and the definition of q_j) we have $c_i w_{ij} q_j^{-1} \leq \frac{1}{\beta n^{3/2}\alpha}$. Therefore

$$||\delta_i''||_2 = \sqrt{\gamma} c_i w_{ij} q_j^{-1} ||y_i - x||_2 \leq \frac{\sqrt{\gamma} d_{max}}{\beta n^{3/2}\alpha}$$

Since $\lim_{n\to\infty} n^{3/2}\alpha = \infty$ by assumption, we have $||\delta_i''||_2 = O(1)$ (this is a loose bound of course, but sufficient for our purposes). Therefore

$$\begin{aligned} \left|\left|\int_{i\notin\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 \leq &\int_{i\notin\hat{N}} p(i)||\delta_i||_2\,\mathrm{d}i + \int_{i\notin\hat{N}} p(i)||\delta_i''||_2\,\mathrm{d}i \\ \leq &\,(d_{max} + O(1)) \int_{i\notin\hat{N}} p(i)\,\mathrm{d}i \end{aligned}$$

By Lemma B.2, $\int_{i\notin\hat{N}} p(i)\,\mathrm{d}i = O(\alpha^{m/2} n^{m/4})$. Since d_{max} is also a constant, we have $\left|\left|\int_{i\notin\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 = O(\alpha^{m/2} n^{m/4})$. □

Lemma D.5. *We have* $\left|\left|n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i\right|\right|_2 \leq \frac{\gamma+\sqrt{\gamma}}{\Omega(n^2\alpha^2)} + O(\alpha^{m/2} n^{m/4+1})$.

Proof. We have

$$\begin{aligned} \left|\left|\int_{i\in N} p(i)\delta_i\,\mathrm{d}i\right|\right|_2 = &\left|\left|\int_{i\in N} p(i)(\delta_i'' - \delta_i'' + \delta_i)\,\mathrm{d}i\right|\right|_2 \\ = &\left|\left|\int_{i\in N} p(i)\delta_i''\,\mathrm{d}i\right|\right|_2 + \left|\left|\int_{i\in N} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 \end{aligned}$$

By Lemma D.1, $\left|\left|\int_{i\in N} p(i)\delta_i''\,\mathrm{d}i\right|\right|_2 = 0$. Thus

$$\begin{aligned} \left|\left|\int_{i\in N} p(i)\delta_i\,\mathrm{d}i\right|\right|_2 = &\left|\left|\int_{i\in N} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 \\ = &\left|\left|\int_{i\in \hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i + \int_{i\notin\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 \\ \leq &\left|\left|\int_{i\in \hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 + \left|\left|\int_{i\notin\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 \end{aligned}$$

where the inequality follows from the triangle inequality of norms. By Lemma D.4, we have $\left|\left|\int_{i\notin\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 = O(\alpha^{m/2} n^{m/4})$, so $\left|\left|\int_{i\in N} p(i)\delta_i\,\mathrm{d}i\right|\right|_2 \leq \left|\left|\int_{i\in\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 + O(\alpha^{m/2} n^{m/4})$. For the next sequence of equations, we will use the triangle inequality, Lemmas D.2 and D.3, and $\int_{i\in\hat{N}} p(i)\,\mathrm{d}i \leq \int_{i\in N} p(i)\,\mathrm{d}i = 1$.

$$\begin{aligned} \left|\left|\int_{i\in\hat{N}} p(i)(\delta_i - \delta_i'')\,\mathrm{d}i\right|\right|_2 = &\left|\left|\int_{i\in\hat{N}} p(i)(\delta_i'' - \delta_i' + \delta_i' - \delta_i)\,\mathrm{d}i\right|\right|_2 \\ \leq &\left|\left|\int_{i\in\hat{N}} p(i)(\delta_i'' - \delta_i')\,\mathrm{d}i\right|\right|_2 + \left|\left|\int_{i\in\hat{N}} p(i)(\delta_i' - \delta_i)\,\mathrm{d}i\right|\right|_2 \\ \leq &\int_{i\in\hat{N}} p(i)||\delta_i'' - \delta_i'||_2\,\mathrm{d}i + \int_{i\in\hat{N}} p(i)||\delta_i' - \delta_i||_2\,\mathrm{d}i \\ \leq &\frac{2\gamma d_{max}}{\beta^2 n^3\alpha^2}\int_{i\in\hat{N}} p(i)\,\mathrm{d}i + \frac{\sqrt{\gamma}}{\beta\eta n^{3/2}\alpha} d_{max}\int_{i\in\hat{N}} p(i)\,\mathrm{d}i \\ = &\frac{2\gamma d_{max}}{\beta^2 n^3\alpha^2} + \frac{\sqrt{\gamma}}{\beta\eta n^{3/2}\alpha} d_{max} \\ = &\frac{\gamma}{\Omega(n^3\alpha^2)} + \frac{\sqrt{\gamma}}{\Omega(n^3\alpha^2)} \end{aligned}$$

Therefore

$$\left|\left|\int_{i\in N} p(i)\delta_i\,\mathrm{d}i\right|\right|_2 \leq \frac{\gamma}{\Omega(n^3\alpha^2)} + \frac{\sqrt{\gamma}}{\Omega(n^3\alpha^2)} + O(\alpha^{m/2} n^{m/4})$$

To obtain the required bound for $\left|\left|n\int_{i\in N} p(i)\delta_i(x)\,\mathrm{d}i\right|\right|_2$, we simply multiply the above expression by n. □

Lemmas B.1, C.6, and D.5 together imply Theorem 4.1.

E Non-triviality of our solution concept

As mentioned in Section 1.1, our definition of an equal-power equal-λ equilibrium may appear circular, since we are maximizing welfare and evaluating λ_i with respect to a utility scale which we get to choose. In Section 1.1, we argued that intuitively, this is analogous to the definition of equality by allocating equal amounts of an artificial currency. In this section, we argue that our solution concept is mathematically nontrivial, by (informally) showing that a particular "obvious" choice for $\mathbf{c}$ is not sufficient. Specifically, we show that a uniform $\mathbf{c}$ (i.e., $c_i = c$ for all i) cannot lead to an equal-power equal-λ equilibrium in general. Making this argument formal would require substantial algebra similar to that in Section C.2, which we feel would obscure the primary intuition.

Recall Lemma C.2, where $q_j = n \int_{k \in N} p(k) c_k w_{kj} \,\mathrm{d}k$:

Lemma C.2. *For every agent i and issue j, $\delta_{ij} = \frac{c_i w_{ij}(y_{ij} - x_j)}{c_i w_{ij} + \lambda_i q_j}$.*

Suppose there exists some $c > 0$ such that $c_i = c$ for all $i \in N$. Then δ_{ij} reduces to

$$\delta_{ij} = \frac{c w_{ij}(y_{ij} - x_j)}{c w_{ij} + \lambda_i c n \int_{k \in N} p(k) w_{kj} \,\mathrm{d}k} = \frac{w_{ij}(y_{ij} - x_j)}{w_{ij} + \lambda_i n \int_{k \in N} p(k) w_{kj} \,\mathrm{d}k}$$

As n goes to infinity, the $\lambda_i n \int_{k \in N} p(k) w_{kj} \,\mathrm{d}k$ term dominates the $c_i w_{ij}$ term. Thus we can approximate δ_{ij} by

$$\delta_{ij} \approx \frac{w_{ij}(y_{ij} - x_j)}{\lambda_i n \int_{k \in N} p(k) w_{kj} \,\mathrm{d}k} = \frac{c w_{ij}(y_{ij} - x_j)}{\lambda_i q_j}$$

Recall that for any agent i whose power constraint is not tight, we have $\lambda_i = 0$; this is a standard property of Lagrange multipliers. Clearly for those agents, we cannot have $(1 - \varepsilon')\lambda \leq \lambda_i \leq \lambda$ for any $\lambda > 0$. An equal-power equal-λ ε'-equilibrium requires that the above hold for at least a $1 - \varepsilon'$ fraction of the agents, so at least a $1 - \varepsilon'$ fraction of the agents must have a tight power constraint. Thus in order to achieve an exact equal-power equal-λ equilibrium as $n \to \infty$, almost all of the agents must have a tight power constraint as $n \to \infty$.

Let i be an arbitrary agent with a tight power constraint. By Lemma C.1, this implies that $\sum_{j \in M} \delta_{ij}^2 q_j = \gamma$. Thus we have

$$\sum_{j \in M} \left(\frac{c w_{ij}(y_{ij} - x_j)}{\lambda_i q_j} \right)^2 q_j \approx \gamma$$

$$\sum_{j \in M} c^2 w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} \approx \lambda_i^2$$

$$c^2 \sum_{j \in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1} \approx \lambda_i^2$$

We need every pair of agents $i, i' \in \tilde{N}$ to satisfy $\lambda_i / \lambda_k \approx 1$. That means we need

$$\frac{\sum_{j \in M} w_{ij}^2 (y_{ij} - x_j)^2 (\int_{k \in N} p(k) c_k w_{kj} \,\mathrm{d}k)^{-1}}{\sum_{j \in M} w_{i'j}^2 (y_{i'j} - x_j)^2 (\int_{k \in N} p(k) c_k w_{kj} \,\mathrm{d}k)^{-1}} \approx 1$$

Note that $(\int_{k \in N} p(k) c_k w_{kj} \,\mathrm{d}k)^{-1}$ is just some constant. Consider a pair of agents i and i' such that $|y_{ij} - x_j| > |y_{i'j} - x_j|$ and $w_{ij} > w_{i'j}$ for each $j \in M$: then this ratio can never approach 1, even as $n \to \infty$. For an appropriate choice of distribution p, this means that there will always be constant measure set of agents whose values of λ_i are not close to the Lagrange multipliers of other agents.

This means that in general, choosing the same scaling factor for each agent will not be sufficient for an equal-power equal-λ equilibrium. Furthermore, the above reasoning intuitively suggests that we really do need $c_i \approx 1/\sqrt{\sum_{j \in M} w_{ij}^2 (y_{ij} - x_j)^2 q_j^{-1}}$ for (almost) every $i \in N$.

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