Plan

✓ Arrow, Sen, Muller-Satterthwaite
✓ Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
✓ Voting to get things “right” (Distance-based measures, Condorcet and extensions)
✓ Strategizing (Gibbard-Satterthwaite)

1. Generalizations
   1.1 Infinite Populations
   ✓ Judgement aggregation (List & Dietrich)

2. Logics

3. Applications
Plan

▶ The logic of axiomatization results
▶ Logics for reasoning about aggregation methods
▶ Preference (modal) logics
▶ Applications
Setting the Stage: Logic and Games


What do the (Im)possibility results say?

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Given a **semantic domain** $\mathcal{D}$ and a **target class** $\mathcal{T} \subseteq \mathcal{D}$
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Fix a language $\mathcal{L}$ and a **satisfaction relation** $\models \subseteq \mathcal{D} \times \mathcal{L}$
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\(\Delta \subseteq \mathcal{L}\) be a set of **axioms**
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Fix a language $\mathcal{L}$ and a **satisfaction relation** $\models \subseteq \mathcal{D} \times \mathcal{L}$

$\Delta \subseteq \mathcal{L}$ be a set of **axioms**

$\Delta$ **absolutely axiomatizes** $\mathcal{T}$ iff for all $M \in \mathcal{D}$, $M \in \mathcal{T}$ iff $M \models \Delta$

(i.e., $\Delta$ **defines** $\mathcal{T}$)
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(i.e., $\Delta$ *defines* $\mathcal{T}$)

$\Delta$ **relatively axiomatizes** $\mathcal{T}$ iff for all $\varphi \in \mathcal{L}$, $\mathcal{T} \models \varphi$ iff $\Delta \models \varphi$

(i.e., $\Delta$ axiomatizes the theory of $\mathcal{T}$)
What do the (Im)possibility results say?

**May’s Theorem:** $\Delta$ is the set of aggregation functions w.r.t. 2 candidates, $\mathcal{T}$ is majority rule, $\mathcal{L}$ is the language of set theory, $\Delta$ is the properties of May’s theorem, then $\Delta$ absolutely axiomatizes $\mathcal{T}$.
What do the (Im)possibility results say?

**May’s Theorem:** $\Delta$ is the set of aggregation functions w.r.t. 2 candidates, $T$ is majority rule, $\mathcal{L}$ is the language of set theory, $\Delta$ is the properties of May’s theorem, then $\Delta$ absolutely axiomatizes $T$.

**Arrow’s Theorem:** $\Delta$ is the set of aggregation functions w.r.t. 3 or more candidates, $T$ is a dictatorship, $\mathcal{L}$ is the language of set theory, $\Delta$ is the properties of May’s theorem, then $\Delta$ absolutely axiomatizes $T$. 
A Minimal Language


Let $\Phi_I$ be the set of **individual formulas** (standard propositional language)

$V_I$ the set of individual valuations
A Minimal Language


Let $\Phi_I$ be the set of **individual formulas** (standard propositional language)

$V_I$ the set of individual valuations

$\Phi_C$ the set of **collective formulas**: $\square \alpha \mid \varphi \land \psi \mid \neg \varphi$

$\square \alpha$: *The group collectively accepts* $\alpha$.

$V_C$ the set of collective valuations: $v : \Phi_C \rightarrow \{0, 1\}$
A Minimal Language

Let \( \mathcal{CON}_n = \{ v \in V_C \mid v(\Box \alpha) = 1 \text{ iff } \forall i \leq n, \; v_i(\alpha) = 1 \} \)

E. \( \Box \varphi \leftrightarrow \Box \psi \) provided \( \varphi \leftrightarrow \psi \) is a tautology

M. \( \Box (\varphi \land \psi) \rightarrow (\Box \varphi \land \Box \psi) \)

C. \( (\Box \varphi \land \Box \psi) \rightarrow (\Box \varphi \land \Box \psi) \)

N. \( \Box \top \)

D. \( \neg \Box \bot \)

**Theorem** [Pauly, 2005] \( V_C(KD) = \mathcal{CON}_n \), provided \( n \geq 2^{\| \Phi_0 \|} \).

\((\mathcal{D} = V_C, \; \mathcal{T} = \mathcal{CON}_n, \; \Delta = EMCND, \text{ then } \Delta \text{ absolutely axiomatizes } \mathcal{T}.)\)
A Minimal Language

Let $\mathcal{MAJ}_n = \{v \in \mathcal{V}_C \mid v([>]\alpha) = 1 \text{ iff } |\{i \mid v_i(\alpha) = 1\}| > \frac{n}{2}\}$

STEM contains all instances of the following schemes

S. $[>]\varphi \rightarrow \neg[>]\neg\varphi$

T. $([\geq]\varphi_1 \land \cdots \land [\geq]\varphi_k \land [\leq]\psi_1 \land \cdots \land [\leq]\psi_k) \rightarrow \land_{1 \leq i \leq k}(=]\varphi_i \land [=]\psi_i)$ where $\forall v \in \mathcal{V}_I:
|\{i \mid v(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

E. $[>]\varphi \leftrightarrow [>]\psi$ provided $\varphi \leftrightarrow \psi$ is a tautology

M. $[>]\varphi \land [>]\psi$ \rightarrow ([>]\varphi \land [>]\psi)$

**Theorem** [Pauly, 2005] $\mathcal{V}_C(STEM) = \mathcal{MAJ}$.

($\mathcal{D} = \mathcal{V}_C$, $\mathcal{T} = \mathcal{MAJ}_n$, $\Delta = STEM$, then $\Delta$ absolutely axiomatizes $\mathcal{T}$.)
Compare principles in terms of the language used to express them


Compare principles in terms of the language used to express them


How much “classical logic” is “needed” for the judgement aggregation results?


Plan

✓ The logic of axiomatization results
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Judgement Aggregation Logic


Some Notation:

- \( N = \{1, \ldots, n\} \) a set of agents
- \( A \) is the agenda (set of formulas of some logic \( \mathcal{L} \) “on the table” satisfying certain “fullness conditions”)
- Let \( J(A, \mathcal{L}) \) is the set of *judgements* (eg. maximally consistent subsets of \( A \))
- \( \gamma \in J(A, \mathcal{L})^n \) is a *judgement profile* with \( \gamma_i \) agent \( i \)’s judgement set
Judgement Aggregation Logic: Semantics

Tables \( \langle F, \gamma, p \rangle \)
Judgement Aggregation Logic: Semantics

Tables $\langle F, \gamma, p \rangle$

Example:

<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Individual 1</td>
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<tr>
<td>$F_{maj}$</td>
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</tbody>
</table>

$A = \{ P, Q, P \rightarrow Q, \neg P, \neg Q, \neg (P \rightarrow Q) \}$

$F$ is an aggregations function $F : J(A, \mathcal{L})^n \rightarrow J(A, \mathcal{L})$
Judgement Aggregation Logic: Semantics

Tables $⟨F, γ, p⟩$

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$γ ∈ J(A, \mathcal{L})^n$ (assuming consistency and completeness)
Judgement Aggregation Logic: Semantics

Tables \( \langle F, \gamma, p \rangle \)

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\[ A = \{ P, Q, P \rightarrow Q, \neg P, \neg Q, \neg (P \rightarrow Q) \} \]

\[ p \in A \]
Judgement Aggregation Logic: Language

Atomic Formulas: $At = \{ i, \sigma, h_p \mid p \in \mathcal{A}, i \in N \}$

Formulas: $\varphi ::= \alpha \mid \Box \varphi \mid \blacklozenge \varphi \mid \varphi \land \varphi \mid \neg \varphi$
Judgement Aggregation Logic: Language
Judgement Aggregation Logic: Truth

- $F, \gamma, p \models h_q$ iff $q = p$

- $F, \gamma, p \models i$ iff $p \in \gamma_i$

- $F, \gamma, p \models \sigma$ iff $p \in F(\gamma)$

- $F, \gamma, p \models \Box \varphi$ iff $\forall \gamma' \in J(A, \mathcal{L})^n, F, \gamma', p \models \varphi$

- $F, \gamma, p \models \blacksquare \varphi$ iff $\forall p' \in A, F, \gamma, p' \models \varphi$

- Boolean connectives as usual
Judgement Aggregation Logic: Truth

- \( F, \gamma, p \models h_q \) iff \( q = p \)
  The current proposition on the table is \( q \)

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- $F, \gamma, p \models h_q$ iff $q = p$
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- $F, \gamma, p \models i$ iff $p \in \gamma_i$
  Voter $i$ accepts the current proposition on the table

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  Society accepts the current proposition on the table

- $F, \gamma, p \models \square \varphi$ iff $\forall \gamma' \in J(A, \mathcal{L})^n, F, \gamma', p \models \varphi$

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- $F, \gamma, p \models \sigma$ iff $p \in F(\gamma)$
  Society accepts the current proposition on the table

- $F, \gamma, p \models \Box \varphi$ iff $\forall \gamma' \in J(A, \mathcal{L})^n$, $F, \gamma', p \models \varphi$
  Quantification over the set of judgement profiles

- $F, \gamma, p \models \lozenge \varphi$ iff $\forall p' \in A$, $F, \gamma, p' \models \varphi$

- Boolean connectives as usual
Judgement Aggregation Logic: Truth

- $F, \gamma, p \models h_q$ iff $q = p$
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- $F, \gamma, p \models \square \varphi$ iff $\forall p' \in A, F, \gamma, p' \models \varphi$
  Quantification over the agenda

- Boolean connectives as usual
### Judgement Aggregation Logic: Example

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\[
\begin{array}{|c|c|c|c|}
\hline
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\[A = \{P, Q, P \rightarrow Q, \neg P, \neg Q, \neg(P \rightarrow Q)\}\]
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$A = \{P, Q, P \rightarrow Q, \neg P, \neg Q, \neg(P \rightarrow Q)\}$

$F_{maj}, \gamma, P \models 1 \land 2 \land \neg 3$
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$F_{maj}, \gamma, P \models \sigma$
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$F_{maj}, \gamma, P \models \Diamond (1 \land 3)$
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$F_{maj}, \gamma, P \models \lozenge((1 \leftrightarrow 2) \land (2 \leftrightarrow 3) \land (1 \leftrightarrow 3))$
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$F_{maj}, \gamma, P \models \diamond((1 \leftrightarrow 2) \land (2 \leftrightarrow 3) \land (1 \leftrightarrow 3))$
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$$\mathcal{A} = \{P, Q, P \rightarrow Q, \neg P, \neg Q, \neg(P \rightarrow Q)\}$$

$$F_{maj}, \gamma, P \models \diamond(\langle 1 \leftrightarrow 2 \rangle \land \langle 2 \leftrightarrow 3 \rangle \land \langle 1 \leftrightarrow 3 \rangle)$$

All agents agree on $P$
Judgement Aggregation Logic: Example

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$\mathcal{A} = \{ P, Q, P \rightarrow Q, \neg P, \neg Q, \neg(P \rightarrow Q) \}$

$F_{maj}, \gamma, P \models \lozenge\blacklozenge((1 \leftrightarrow 2) \land (2 \leftrightarrow 3) \land (1 \leftrightarrow 3))$
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\[ F_{maj}, \gamma, P \models \Diamond \Box ((1 \leftrightarrow 2) \land (2 \leftrightarrow 3) \land (1 \leftrightarrow 3)) \]
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$A = \{P, Q, P \to Q, \neg P, \neg Q, \neg(P \to Q)\}$

$F_{maj}, \gamma, P \models \Diamond ((1 \leftrightarrow 2) \land (2 \leftrightarrow 3) \land (1 \leftrightarrow 3))$

All agents agree on all propositions in the agenda
$$F_{maj}, \gamma, P \models \Box\Box (\sigma \leftrightarrow \bigvee_{G \subseteq \{1,2,3\}, |G| \geq 2} \land_{i \in G} i)$$
Judgement Aggregation Logic: Results

Sound and complete axiomatization

Model checking is decidable, but relatively difficult

Expressivity:

• Discursive Dilemma:♦
  \( \text{MV} := \sigma \leftrightarrow \bigvee G \subseteq N, |G| > n^2 \land i \in G \)

• Impossibility results:
  Nondictatorship: ♦♦
  \( \neg (\sigma \leftrightarrow i) \)
  Unanimity: □■
  \( (1 \land \cdots \land n) \rightarrow \sigma \)
  Independence: □
  \( o \in O \Box (o \land \sigma \rightarrow \Box (o \rightarrow \sigma)) \)

Given any judgement profile, any choice of the voters and any \( P \in A \), if society accepts \( P \) then for any profile (if the choices are the same w.r.t. \( P \) then society should accept \( P \))
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  - Unanimity: $\Box ((1 \land \cdots \land n) \rightarrow \sigma)$
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- Complete axiomatization
Plan

✓ The logic of axiomatization results
✓ Logics for reasoning about aggregation methods
  ▶ Preference (modal) logics
  ▶ Applications
Preference (Modal) Logics

\(x, y\) objects

\(x \succeq y: x\) is at least as good as \(y\)
Preference (Modal) Logics

$x, y$ objects

$x \succeq y$: $x$ is at least as good as $y$

1. $x \succeq y$ and $y \not\succeq x$ ($x \succ y$)
2. $x \not\succeq y$ and $y \succeq x$ ($y \succ x$)
3. $x \succeq y$ and $y \succeq x$ ($x \sim y$)
4. $x \not\succeq y$ and $y \not\succeq x$ ($x \perp y$)
Preference (Modal) Logics

$x, y$ objects

$x \succeq y$: $x$ is at least as good as $y$

1. $x \succeq y$ and $y \not\succeq x$ ($x \succ y$)
2. $x \not\succeq y$ and $y \succeq x$ ($y \succ x$)
3. $x \succeq y$ and $y \succeq x$ ($x \sim y$)
4. $x \not\succeq y$ and $y \not\succeq x$ ($x \perp y$)

Properties: transitivity, connectedness, etc.
Preference (Modal) Logics

Modal betterness model $\mathcal{M} = \langle W, \preceq, V \rangle$
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Modal betterness model $\mathcal{M} = \langle W, \succeq, V \rangle$

Preference Modalities $\langle \succeq \rangle \varphi$: “there is a world at least as good (as the current world) satisfying $\varphi$”

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$\mathcal{M}, w \models \langle \succ \rangle \varphi$ iff there is $v \succeq w$ and $w \not\succeq v$ such that $\mathcal{M}, v \models \varphi$
Preference (Modal) Logics

1. $\langle \neg \rangle \varphi \rightarrow \langle \geq \rangle \varphi$
2. $\langle \leq \rangle \langle \neg \rangle \varphi \rightarrow \langle \neg \rangle \varphi$
3. $\varphi \land \langle \leq \rangle \psi \rightarrow (\langle \neg \rangle \psi \lor \langle \leq \rangle (\psi \land \langle \leq \rangle \varphi))$
4. $\langle \neg \rangle \langle \leq \rangle \varphi \rightarrow \langle \neg \rangle \varphi$

**Theorem** The above logic (with Necessitation and Modus Ponens) is sound and complete with respect to the class of preference models.

Preference Modalities

$\varphi \geq \psi$: the state of affairs $\varphi$ is at least as good as $\psi$ (ceteris paribus)

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$$\varphi \geq \psi:$$ the state of affairs $$\varphi$$ is at least as good as $$\psi$$ (ceteris paribus)


$$\langle \Gamma \rangle \leq \varphi:$$ $$\varphi$$ is true in “better” world, *all things being equal*.

All Things Being Equal...

With boots ($b$), I prefer my raincoat ($r$) over my umbrella ($u$).

Without boots ($\neg b$), I also prefer my raincoat ($r$) over my umbrella ($u$).

But I do prefer an umbrella and boots over a raincoat and no boots.
All Things Being Equal...

- With boots ($b$), I prefer my raincoat ($r$) over my umbrella ($u$).

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Eric Pacuit: The Logic Behind Voting
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All Things Being Equal...

All things being equal, I prefer my raincoat over my umbrella.
All Things Being Equal...

Let $\Gamma$ be a set of (preference) formulas. Write $w \equiv_\Gamma v$ if for all $\varphi \in \Gamma$, $w \models \varphi$ iff $v \models \varphi$.

1. $\mathcal{M}, w \models \langle \Gamma \rangle \varphi$ iff there is a $v \in W$ such that $w \equiv_\Gamma v$ and $\mathcal{M}, v \models \varphi$.

2. $\mathcal{M}, w \models \langle \Gamma \rangle \leq \varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap \leq)v$ and $\mathcal{M}, v \models \varphi$.

3. $\mathcal{M}, w \models \langle \Gamma \rangle < \varphi$ iff there is a $v \in W$ such that $w(\equiv_\Gamma \cap <)v$ and $\mathcal{M}, v \models \varphi$. 
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Key Principles:

- $\langle \Gamma' \rangle \varphi \rightarrow \langle \Gamma \rangle \varphi$ if $\Gamma \subseteq \Gamma'$
- $\pm \varphi \land \langle \Gamma \rangle (\alpha \land \pm \varphi) \rightarrow \langle \Gamma \cup \{\varphi\} \rangle \alpha$
All Things Being Equal...

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Key Principles:

$\langle \Gamma' \rangle \leq \varphi \rightarrow \langle \Gamma \rangle \leq \varphi$ if $\Gamma \subseteq \Gamma'$

$\pm \varphi \land \langle \Gamma \rangle \leq (\alpha \land \pm \varphi) \rightarrow \langle \Gamma \cup \{\varphi\} \rangle \leq \alpha$
Preference Lifting, I

Given a preference ordering $\preceq$ over a set of objects $X$, we want to lift this to an ordering $\hat{\preceq}$ over $\mathcal{P}(X)$.

Given $\preceq$, what reasonable properties can we infer about $\hat{\preceq}$?

You know that $x \prec y \prec z$
Can you infer that $\{x, y\} \hat{\prec} \{z\}$?
You know that $x \prec y \prec z$
Can you infer that $\{x, y\} \asymp \{z\}$?

You know that $x \prec y \prec z$
Can you infer anything about $\{y\}$ and $\{x, z\}$?
Preference Lifting, II

- You know that $x \preceq y \preceq z$
  
  Can you infer that $\{x, y\} \sim \{z\}$?

- You know that $x \preceq y \preceq z$
  
  Can you infer anything about $\{y\}$ and $\{x, z\}$?

- You know that $w \preceq x \preceq y \preceq z$
  
  Can you infer that $\{w, x, y\} \sim \{w, y, z\}$?
Preference Lifting, II

▶ You know that $x \prec y \prec z$
Can you infer that $\{x, y\} \equiv \{z\}$?

▶ You know that $x \prec y \prec z$
Can you infer anything about $\{y\}$ and $\{x, z\}$?

▶ You know that $w \prec x \prec y \prec z$
Can you infer that $\{w, x, y\} \equiv \{w, y, z\}$?

▶ You know that $w \prec x \prec y \prec z$
Can you infer that $\{w, x\} \equiv \{y, z\}$?
Preference Lifting, III

There are different interpretations of $X \lesssim Y$:

- You will get one of the elements, but cannot control which.
- You can choose one of the elements.
- You will get the full set.
Preference Lifting, IV

Kelly Principle

(EXT) \{x\} \succ \{y\} provided \(x \prec y\)

(MAX) \(A \succ \text{Max}(A)\)

(MIN) \(\text{Min}(A) \succ A\)

Preference Lifting, IV

Gärdenfors Principle

\( (G1) \) \( A \overset{\sim}{\prec} A \cup \{x\} \) if \( a \prec x \) for all \( a \in A \)

\( (G2) \) \( A \cup \{x\} \overset{\sim}{\prec} A \) if \( x \prec a \) for all \( a \in A \)

Preference Lifting, IV

Gärdenfors Principle

\[(G1) \quad A \hat{\succ} A \cup \{x\} \text{ if } a \prec x \text{ for all } a \in A\]
\[(G2) \quad A \cup \{x\} \hat{\succ} A \text{ if } x \prec a \text{ for all } a \in A\]


Independence

\[(IND) \quad A \cup \{x\} \hat{\preceq} B \cup \{x\} \text{ if } A \hat{\preceq} B \text{ and } x \notin A \cup B\]
Preference Lifting, V

**Theorem** (Kannai and Peleg). If $|X| \geq 6$, then no weak order satisfies both the Gärdenfors principle and independence.

From Worlds to Sets, I

\[ \mathcal{M}, w \models \varphi \preceq \exists \exists \psi \text{ iff there is } s, t \text{ such that } \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, t \models \psi \text{ and } s \preceq t \]
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\( \mathcal{M}, w \models \varphi \preceq \forall \exists \psi \) iff for all \( s \) there is a \( t \) such that \( \mathcal{M}, s \models \varphi \) implies \( \mathcal{M}, t \models \psi \), and \( s \preceq t \)
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From Worlds to Sets, II

\[ \varphi \leq \exists \exists \psi \; := \; E ( \varphi \land \lozenge \leq \psi ) \]
From Worlds to Sets, II

\[ \varphi \preceq \exists \exists \psi := E(\varphi \land \lozenge \preceq \psi) \]

\[ \varphi \prec \exists \exists \psi := E(\varphi \land \lozenge \prec \psi) \]
From Worlds to Sets, II

\[ \varphi \preceq \exists \psi := E(\varphi \land \lozenge \lessdot \psi) \]

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\[ \varphi \preceq \forall \psi := A(\varphi \rightarrow \lozenge \lessdot \psi) \]
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\[ M, w \models \varphi \preceq \forall \forall \psi \text{ iff for all } s, \text{ for all } t, M, s \models \varphi \text{ and } M, t \models \psi \text{ implies } s \preceq t \]
From Worlds to Sets, III

\[ M, w \models \varphi \preceq_{\forall} \psi \text{ iff for all } s, \text{ for all } t, M, s \models \varphi \text{ and } M, t \models \psi \]
implies \( s \preceq t \)

\[ M, w \models \varphi \prec_{\forall} \psi \text{ iff for all } s, \text{ for all } t, M, s \models \varphi \text{ and } M, t \models \psi \]
implies \( s \prec t \)
We must assume the ordering $\preceq$ is total.
\[ \varphi \preceq \forall \psi \colon : A(\psi \rightarrow \Box \preceq \neg \varphi) \]

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We must assume the ordering $\preceq$ is total
From Sets to Worlds

\[ P_1 \gg P_2 \gg P_3 \gg \cdots \gg P_n \]

\( x > y \) iff \( x \) and \( y \) differ in at least one \( P_i \) and the first \( P_i \) where this happens is one with \( P_i x \) and \( \neg P_i y \)

Logics of Knowledge and Preference

\[ K(\varphi \succeq \psi) : \text{“Ann knows that } \varphi \text{ is at least as good as } \psi \text{”} \]

\[ K \varphi \succeq K \psi : \text{“knowing } \varphi \text{ is at least as good as knowing } \psi \]
Logics of Knowledge and Preference

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\( K\varphi \succeq K\psi \): “knowing \( \varphi \) is at least as good as knowing \( \psi \”

\( \mathcal{M} = \langle W, \sim, \succeq, V \rangle \)
Logics of Knowledge and Preference

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\[ \mathcal{M} = \langle W, \sim, \succeq, V \rangle \]


$A(\psi \rightarrow \langle \succeq \rangle \varphi) \ vs. \ K(\psi \rightarrow \langle \succeq \rangle \varphi)$
\[A(\psi \rightarrow \langle \succeq \rangle \varphi) \text{ vs. } K(\psi \rightarrow \langle \succeq \rangle \varphi)\]

Should preferences be restricted to information sets?
$A(\psi \rightarrow \langle \succeq \rangle \varphi) \text{ vs. } K(\psi \rightarrow \langle \succeq \rangle \varphi)$

Should preferences be restricted to information sets?

$\mathcal{M}, w \models \langle \succeq \cap \sim \rangle \varphi$ iff there is a $v$ with $w \sim v$ and $w \preceq v$ such that $\mathcal{M}, v \models \varphi$

$K(\psi \rightarrow \langle \succeq \cap \sim \rangle \varphi)$
\[ \varphi \succeq_X \psi \quad \text{“The agent considers } \varphi \text{ at least as good as } \psi \text{ for reason } X \text{”} \]
\[ \varphi \preceq_X \psi \] “The agent considers \( \varphi \) at least as good as \( \psi \) for reason \( X \)”

i envisions a situation in which \( \varphi \) is true and that otherwise differs little from his actual situation. Likewise i envisions a world where \( \psi \) is true and otherwise differs little from his actual situation. Finally, there utility according to \( u_X \) of the first imagined situation exceeds that of the second.
$p$: “$i$ purchases a fire alarm”
p: “i purchases a fire alarm”

\[ p \succ_1 \neg p: \ u_1 \text{ measures safety} \]
$p$: “$i$ purchases a fire alarm”

$p \succ_1 \neg p$: $u_1$ measures safety

$p \prec_2 \neg p$: $u_2$ measures finances
$p$: “$i$ purchases a fire alarm”

$p \succ_1 \neg p$: $u_1$ measures safety

$p \prec_2 \neg p$: $u_2$ measures finances

What is the status of $p \succ_{1,2} \neg p$? $p \prec_{1,2} \neg p$?
\((\rho \succ_1 \top) \succ_2 \top\): it’s in your financial interest that your buying a low-power automobile is in you safety interesting — which might well be true inasmuch as low-power vehicles are cheaper.
$(p \succ_1 \top) \succ_2 \top$: it’s in your financial interest that your buying a low-power automobile is in you safety interesting — which might well be true inasmuch as low-power vehicles are cheaper.

$\neg q \succ_1 (p \succ_2 q)$: from the point of view of family pride, you’d rather that your brother not run for mayor than that Miss Smith be the superior candidate.
At a set of atomic proposition, \( \mathcal{S} \) a set of reasons.

\[ \langle \mathcal{W}, s, u, V \rangle \]

- \( \mathcal{W} \) is a set of states
- \( s : \mathcal{W} \times \mathcal{S} \neq \emptyset (\mathcal{W}) \rightarrow \mathcal{W} \) is a selection function \((s(w, A) \in A)\)
- \( u : \mathcal{W} \times \mathcal{S} \rightarrow \mathbb{R} \) is a utility function
- \( V : \text{At} \rightarrow \mathcal{S}(\mathcal{W}) \) is a valuation function
At a set of atomic proposition, $\mathcal{S}$ a set of reasons.

$$\langle \mathcal{W}, s, u, V \rangle$$

- $\mathcal{W}$ is a set of states
- $s: \mathcal{W} \times \not=\emptyset(\mathcal{W}) \rightarrow \mathcal{W}$ is a selection function ($s(w, A) \in A$)
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- $V: \text{At} \rightarrow \not=\emptyset(\mathcal{W})$ is a valuation function

$$\mathcal{M}, w \models \theta \succeq_{X} \psi \iff u_{X}(s(w, \llbracket \theta \rrbracket_{\mathcal{M}})) \geq u_{X}(s(w, \llbracket \psi \rrbracket_{\mathcal{M}}))$$

provided $\llbracket \theta \rrbracket_{\mathcal{M}} \neq \emptyset$ and $\llbracket \psi \rrbracket_{\mathcal{M}} \neq \emptyset$
$$\Diamond \varphi = \text{def } \varphi \succeq x \varphi$$
$$\square \varphi = \text{def } \neg (\neg \varphi \succeq x \neg \varphi)$$
Reflexive: for all $w$ if $w \in A$ then $s(w, A) = w$. 

$M$ is reflexive implies $(p \preceq X \top) \lor (\neg p \preceq X \top)$ is valid.

□ $(p \rightarrow (p \prec X \neg p)) \land □ (\neg p \rightarrow (\neg p \succ X p))$
Reflexive: for all $w$ if $w \in A$ then $s(w, A) = w$.

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Reflexive: for all \( w \) if \( w \in A \) then \( s(w, A) = w \).

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\[ \square(p \rightarrow (p \prec_X \neg p)) \land \square(\neg p \rightarrow (\neg p \succ_X p)) \]
Regular: if $A \subseteq B$ and $w_1 \in A$ then \( s(w, B) = w_1 \) then
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$\mathcal{M}$ is regular implies $((p \lor q) \succ X r) \rightarrow ((p \succ X r) \lor (q \succ X r))$ is valid.
$\mathcal{M}$ is regular and reflexive then

$((p \prec_1 \top) \succ_2 (q \prec_1 \top)) \rightarrow (\neg p \succ_2 \neg q)$ is valid.

“If it is ecologically better for $p$ than for $q$ to politically backfire the abstaining from $p$ is ecologically better than abstaining from $q$. ”
\( \mathcal{M} \) is proximal if for all \( w \) and \( A \neq \emptyset \), if \( s(w, A) = w_1 \) then there is no \( w_2 \in A \) such that \( V^{-1}(w) \Delta V^{-1}(w_2) \subset V^{-1}(w) \Delta V^{-1}(w_1) \), where \( \Delta \) is the symmetric difference.
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$$(((p \land r) \succ_X (q \land r)) \land ((p \land \neg r) \succ_X (q \land \neg r))) \rightarrow (p \succ_X q)$$ is invalid in the class of regular and in the class of proximal models, but valid in the class of models that are both proximal and regular.
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\[
(((p \land r) \gg_X (q \land r)) \land ((p \land \neg r) \gg_X (q \land \neg r))) \rightarrow (p \gg_X q)
\]
is invalid in the class of regular and in the class of proximal models, but valid in the class of models that are both proximal and regular.

\[
(p \land ((p \land q) \gg_X r)) \rightarrow (q \gg_X r)
\]
Plan

✓ The logic of axiomatization results
✓ Logics for reasoning about aggregation methods
✓ Preference (modal) logics
  ▶ Applications
Infinite Voting Populations

Given an aggregation method $F$, let $\mathcal{D} = \{ C \mid C \text{ is winning for } F \}$
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Given a set of winning coalitions $\mathcal{D}$, we can define $F$ as follows:

$$F(J) = \{ \alpha \mid \{ i \mid i \text{ judges that } \alpha \} \in \mathcal{D} \}$$
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What is the general relationship between sets of coalitions and aggregators?
Infinite Voting Populations


Theorem. Let $\mathcal{D}$ be a filter and suppose that $F_{\mathcal{D}}$ preserves $\psi$ and assume that there is some $A \in \Omega^I$ with finite witness multiplicity with respect to $\psi$. Then,

- If $\mathcal{D}$ is an ultrafilter, then it is principal (whence $F_{\mathcal{D}}$ is a dictatorship)
- If $\varphi$ is free of negation, disjunction and universal quantification then $\mathcal{D}$ contains a finite coalition (whence $F_{\mathcal{D}}$ is an oligarchy)
May’s Theorem: Notation

Fix an infinite set $\mathcal{W}$.

Suppose that there are two alternatives, $x$ and $y$, under consideration.

We assume that each voter has a linear preference over $x$ and $y$, so for each $w \in \mathcal{W}$, either $w$ prefers $x$ to $y$ or $y$ to $x$, but not both.

Assume that a subset $X \subseteq \mathcal{W}$, represents the set of all voters that prefer $x$ to $y$.

Thus $X$ represents the outcome of a particular vote.
May’s Theorem: Notation

There are three possible outcomes to consider: 0 means that alternative $y$ was chosen, $\frac{1}{2}$ means the vote was a tie, and 1 means that alternative $x$ was chosen.

An aggregation function is a function $f : 2^W \rightarrow \{0, \frac{1}{2}, 1\}$.

A set $X \subseteq W$, $f(X)$ represents the social preference of the group $W$ ($\frac{1}{2}$ is interpreted as a tie).
Properties of $f$

Consider $f : 2^W \rightarrow \{0, \frac{1}{2}, 1\}$
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**Decisiveness** $f$ is a total function.

**Neutrality** for all $X \subseteq W$, $f(X^C) = 1 - f(X)$

**Positive Responsiveness** if, for all $X, Y \subseteq W$, $X \subset Y$ and $f(X) \neq 0$ implies $f(Y) = 1$. 
Anonymity

Anonymity states that it is the number of votes that counts when determining the outcome, not who voted for what.
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When $W$ is finite, this condition is straightforward to impose:

Fix an arbitrary order on $W$, then each subset of $W$ can be represented by a finite sequence of 1s and 0s.

Then $f$ satisfies **anonymity** if $f$ is symmetric in this sequence of 1s and 0s.
Anonymity for an Infinite Population

A permutation on a set $X$ is a 1-1 map $\pi : X \rightarrow X$.

$f$ is anonymous iff for all $\pi$ and $X \subseteq W$, $f(X) = f(\pi[X])$. 
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$f$ is anonymous iff for all $\pi$ and $X \subseteq W$, $f(X) = f(\pi[X])$.

Too strong! Let $X$, $Y$ be any (countably) infinite subsets of $W$, then there is a $\pi$ such that $\pi[X] = Y$. Hence, for all $X$, $Y \subseteq W$, $f(X) = f(Y)$. 
Anonymity for an Infinite Population

A finite permutation on a set $X$ is a 1-1 map $\pi : X \to X$ such that there is a finite set $F \subseteq X$ such that for all $w \in W - F$, $\pi(w) = w$.

$f$ is finitely anonymous iff for all finite permutations $\pi$ and $X \subseteq W$, $f(X) = f(\pi[X])$. 
Let $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $X(n) = \{ m \in X \mid m \leq n \}$

$$d(X) = \lim_{n \to \infty} \frac{X(n)}{n}$$
Digression: Bounded Anonymity and Density

Let $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $X(n) = \{m \in X \mid m \leq n\}$

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Let $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $X(n) = \{ m \in X \mid m \leq n \}$

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$$d(\mathbb{E}) = \frac{1}{2}$$

Unfortunately, $\lim_{n \to \infty} \frac{X(n)}{n}$ does not always exist.

$\pi$ is a bounded permutation iff

$$\lim_{n \to \infty} \frac{\{ k \mid k \leq n < \pi(k) \}}{n} = 0$$
May’s Theorem Generalized

**Bounded anonymity:** \( F(A) = F(\pi[A]) \) for all bounded permutations

**Density positive responsiveness:** \( f \) satisfies monotonicity and, if \( f(A) = 1/2 \) and all sets with density \( D \) with \( A \cap D \neq \emptyset \) and \( d(A) > 1 \), we have \( f(A \cup D) = 1 \).

**Theorem** (Fey) If an aggregation rule \( f \) satisfies neutrality, density positive responsiveness and bounded anonymity, then \( f \) agrees with a density majority rule.

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Broader Applications

Is it possible to choose rationally among rival scientific theories on the basis of the accuracy, simplicity, scope and other relevant criteria? No

Broader Applications

- Is it possible to choose rationally among rival scientific theories on the basis of the accuracy, simplicity, scope and other relevant criteria? No / Yes


M. Moureau. *Mr. Accuracy, Mr. Simplicity and Mr. Scope: from social choice to theory choice*. FEW, 2012.
Broader Applications

▶ Is it possible to choose rationally among rival scientific theories on the basis of the accuracy, simplicity, scope and other relevant criteria? No/Yes


M. Moureau. *Mr. Accuracy, Mr. Simplicity and Mr. Scope: from social choice to theory choice*. FEW, 2012.

▶ Is it possible to rationally merge evidence from multiple methods?

Broader Applications

Is it possible to merge classic AGM belief revision with the Ramsey test?


Plan

✓ Arrow, Sen, Muller-Satterthwaite
✓ Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
✓ Voting to get things “right” (Distance-based measures, Condorcet and extensions)
✓ Strategizing (Gibbard-Satterthwaite)
✓ Generalizations
  ✓ Infinite Populations
  ✓ Judgement aggregation (List & Dietrich)
✓ Logics
✓ Applications
Thank you!