Plan for the Course

✓ Introduction, Motivation and Basic Epistemic Logic

Lecture 2: Other models of Knowledge, Knowledge in Groups and Group Knowledge

Lecture 3: Reasoning about Knowledge and .......

Lecture 4: Logical Omniscience and Other Problems

Lecture 5: Reasoning about Knowledge in the Context of Social Software
Epistemic Logic

The Language: $\varphi := p | \neg \varphi | \varphi \land \psi | K\varphi$

Kripke Models: $\mathcal{M} = \langle W, R, V \rangle$ with $R$ an equivalence relation

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \text{At}$)
- $\mathcal{M}, w \models \neg \varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \land \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K\varphi$ if for each $v \in W$, if $wRv$, then $\mathcal{M}, v \models \varphi$
Epistemic Logic

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Epistemic Logic

The Language: \( \varphi := p \mid \neg \varphi \mid \varphi \land \psi \mid K_i \varphi \) with \( i \in A \)

Kripke Models: \( \mathcal{M} = \langle W, \{R_i\}_{i \in A}, V \rangle \) with \( R_i \) and equivalence relation.

Truth: \( \mathcal{M}, w \models \varphi \) is defined as follows:

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4. \( \mathcal{M}, w \models K_i \varphi \) if for each \( v \in W \), if \( w R_i v \), then \( \mathcal{M}, v \models \varphi \)
## Results

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The logic **S5** contains the following axioms and rules:

\[
\begin{align*}
&\text{Pc} & \text{Axiomatization of Propositional Calculus} \\
&K & K(\varphi \to \psi) \to (K\varphi \to K\psi) \\
&T & K\varphi \to \varphi \\
&4 & K\varphi \to KK\varphi \\
&5 & \neg K\varphi \to K\neg K\varphi \\
&\text{MP} & \frac{\varphi \varphi \to \psi}{\psi} \\
&\text{Nec} & \frac{\varphi}{K\psi}
\end{align*}
\]

**Theorem**

**S5** is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.
Other Models

- Other Models
- Aumann Structures
- Group Knowledge
Other Models

Let $W$ be a set of worlds, or states.
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Let $W$ be a set of worlds, or states.

Let $S$ be the set of all states of nature.

A set $E \subseteq W$, called an event, is true at state $w$ if $w \in E$. 
Definition
Aumann Model An **Aumann model based on** $S$ is a triple $\langle W, \Pi, \sigma \rangle$, where $W$ is a nonempty set, $\Pi$ is a partition over $W$ and $\sigma : W \rightarrow S$.

Definition
Knowledge Function Let $M = \langle W, \Pi, \sigma \rangle$ be an Aumann model. The **knowledge function**, $K : \wp(W) \rightarrow \wp(W)$, based on $M$ is defined as follows:

$$K(E) = \{ w | \Pi(w) \subseteq E \}$$
Lemma

Let $\mathcal{M} = \langle W, \Pi, \sigma \rangle$ be a Aumann model and $K$ the knowledge function based on $\mathcal{M}$. For each $E, F \subseteq W$

- $E \subseteq F \implies K(E) \subseteq K(F)$ \hspace{1cm} \text{Monotonicity}
- $K(E \cap F) = K(E) \cap K(F)$ \hspace{1cm} \text{Closure Under Intersection}
- $K(E) \subseteq E$ \hspace{1cm} \text{Truth}
- $K(E) \subseteq K(K(E))$ \hspace{1cm} \text{Positive introspection}
- $\overline{K(E)} \subseteq \overline{K(K(E))}$ \hspace{1cm} \text{Negative introspection}
- $K(\emptyset) = \emptyset$ \hspace{1cm} \text{Consistency}

where $\overline{E}$ means the set-theoretic complement of $E$ (relative to $W$).
We can give analogous correspondence, completeness, etc. proofs.
Let $W$ be a set of worlds and $\Delta(W)$ be the set of probability distributions over $W$.

We are interested in functions $p : W \rightarrow \Delta(W)$.

The basic intuition is that for each state $w \in W$, $p(w) \in \Delta(W)$ is a probability function over $W$.

So, $p(w)(v)$ is the probability the agent assigns to state $v$ in state $w$. To ease notation we write $p_w$ for $p(w)$. 
Definition
The pair $\langle W, p \rangle$ is called a **Bayesian frame**, where $W \neq \emptyset$ is any set, and $p : W \rightarrow \Delta(W)$ is a function such that

\[
\text{if } p_w(v) > 0 \text{ then } p_w = p_v
\]

Given a Bayesian frame $\mathcal{F} = \langle W, p \rangle$ and a set of states $S$, an **Bayesian model based on S** is a triple $\langle W, p, \sigma \rangle$, where $\sigma : W \rightarrow S$.

Definition
For each $r \in [0, 1]$ define $B^r : 2^W \rightarrow 2^W$ as follows

\[
B^r(E) = \{ w \mid p_w(E) \geq r \}
\]
Observation: We can define a possibility model from a Bayesian model as follows. Let $\langle W, p, \sigma \rangle$ be a Bayesian model on a state space $S$. We define a possibility model $\langle W, P, \sigma \rangle$ base on $S$ as follows: define $P : W \rightarrow 2^W$ by

$$P(w) = \{ v \mid \pi_w(v) > 0 \}$$

It is easy to see that $P$ is serial, transitive and Euclidean.
Group Knowledge

- Other Models

- Aumann Structures

- Group Knowledge
Suppose there are two friends Ann and Bob are on a bus separated by a crowd. Before the bus comes to the next stop a mutual friend from outside the bus yells “get off at the next stop to get a drink?”.

Say Ann is standing near the front door and Bob near the back door. When the bus comes to a stop, will they get off?


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Three Views of Common Knowledge

1. \( \gamma := i \text{ knows that } \varphi, \ j \text{ knows that } \varphi, \ i \text{ knows that } j \text{ knows that } \varphi, \ j \text{ knows that } i \text{ knows that } \varphi, \ i \text{ knows that } j \text{ knows that } i \text{ knows that } \varphi, \ldots \)


2. \( \gamma := \ i \text{ and } j \text{ know that } (\varphi \text{ and } \gamma) \)


3. There is a *shared situation* \( s \) such that
   - \( s \) entails \( \varphi \)
   - \( s \) entails \( i \text{ knows } \varphi \)
   - \( s \) entails \( j \text{ knows } \varphi \)


Three Views of Common Knowledge

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2. $\gamma := i$ and $j$ know that $(\varphi$ and $\gamma)$


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Recall $K_i : \wp(W) \rightarrow \wp(W)$ is $i$’s knowledge function.

Define $K^m : \wp(W) \rightarrow \wp(W)$ for $m \geq 1$ by

- $K^1 E := \bigcap_{i \in A} K_i E$
- $K^{m+1} E := K^1(K^m(E))$

$K^1 E$ means everyone knows $E$

$K^2 E$ means everyone knows that everyone knows $E$

Define $K^\infty : \wp(W) \rightarrow \wp(W)$

$$K^\infty E := K^1 E \cap K^2 E \cap \cdots \cap K^m E \cap \cdots$$
Common Knowledge: Iterated View

Recall $K_i : \wp(W) \rightarrow \wp(W)$ is $i$’s knowledge function.

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K^\infty E := K^1 E \cap K^2 E \cap \ldots \cap K^m E \cap \ldots
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**Fact** Prove that for all $i \in A$ and $E \subseteq W$, $K_i K^\infty(E) = K^\infty(E)$. 
**Fact** Prove that for all \( i \in \mathcal{A} \) and \( E \subseteq W \), \( K_i K^\infty(E) = K^\infty(E) \).

Suppose you are told “Ann and Bob are going together,”’ and respond “sure, that’s common knowledge.” What you mean is not only that everyone knows this, but also that the announcement is pointless, occasions no surprise, reveals nothing new; in effect, that the situation after the announcement does not differ from that before.

...the event “Ann and Bob are going together” — call it \( E \) — is common knowledge if and only if some event — call it \( F \) — happened that entails \( E \) and also entails all players’ knowing \( F \) (like all players met Ann and Bob at an intimate party). (Aumann, pg. 271, footnote 8)
Definition
Self-Evident Event An event $F$ is **self-evident** if $K_i(F) = F$ for all $i \in \mathcal{A}$.

Definition
Knowledge Field Let $\langle \mathcal{W}, \{\Pi_i\}_{i \in \mathcal{A}}, \sigma \rangle$ be a multi-agent Aumann model. For each $i \in \mathcal{A}$, the **knowledge field of** $i$, denoted $\text{K}_i$, is the family of all unions of cells in $\Pi_i$.

Lemma
An event $E$ is commonly known iff some self-evident event that entails $E$ obtains. Formally, $K^{\infty}(E)$ is the largest event in $\bigcap_{i \in \mathcal{A}} \text{K}_i$ that is included in $E$. 
Common Knowledge in Epistemic Logic

Definition

The operator “everyone knows $\varphi$”, denoted $E\varphi$, is defined as follows

$$E\varphi := \bigwedge_{i \in A} K_i \varphi$$
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Definition
The multi-agent epistemic language with common knowledge is generated by the following grammar:

$$ p \mid \neg \varphi \mid \varphi \land \varphi \mid K_i \varphi \mid C \varphi $$

where $p \in \text{At}$ and $i \in A$. 
Common Knowledge in Epistemic Logic

Definition
The truth of $C \varphi$ is:

$$M, w \models C \varphi \text{ iff } \text{for all } v \in W, \text{ if } w R^* v \text{ then } M, v \models \varphi$$

where $R^* := (\bigcup_{i \in A} R_i)^*$ is the reflexive transitive closure of the union of the $R_i$’s.
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$$\mathcal{M}, w \models C\varphi \text{ iff every finite path starting at } w \text{ ends with a state satisfying } \varphi.$$
Example

$P$ means “The talk is at 2PM”.
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$\mathcal{M}, s \models \neg KB P$
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Group Knowledge

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Another Example

Two players Ann and Bob are told that the following will happen. Some positive integer \( n \) will be chosen and one of \( n, n + 1 \) will be written on Ann’s forehead, the other on Bob’s. Each will be able to see the other’s forehead, but not his/her own.

Suppose the number are (2,3).

Do the agents know there numbers are less than 1000?

Is it common knowledge that their numbers are less than 1000?
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Logics with Common Knowledge

The following axiom and rule need to be added to $S5$ to deal with the common knowledge operator:

- $C(\varphi \rightarrow \psi) \rightarrow (C\varphi \rightarrow C\psi)$
- $C\varphi \rightarrow (\varphi \land EC\varphi)$
- $C(\varphi \rightarrow E\varphi) \rightarrow (\varphi \rightarrow C\varphi)$

Theorem

$S5^C$ is sound and weakly complete with respect to the class of all Kripke frames where the relations are equivalence relations.
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**Theorem**

$S5^C$ is sound and weakly complete with respect to the class of all Kripke frames where the relations are equivalence relations.
In Relational Models (including Aumann Structures), the first two views of Common Knowledge are mathematically equivalent and it is not clear how to represent the third view.

They can be separated using alternative semantics.


There are other notions of “group knowledge”: distributed knowledge

In many social situations, other levels of knowledge are of interest...
Concluding Remark

- In Relational Models (including Aumann Structures), the first two views of Common Knowledge are mathematically equivalent and it is not clear how to represent the third view.

- They can be separated using alternative semantics.
  
  

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Time for a beak.